Representations of direct sums

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Abstract

We investigate the representation theory of direct sums of Lie algebras. In particular, we focus on the representation theory of $\mathfrak{so}(4) \simeq \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$.

Keywords— representation theory

1 General theory

In order to understand the representation theory of direct sums of Lie algebras, we need to understand the irreducible representations. We begin with the following example of a representation of a direct sum.

Definition 1.1. *Suppose $\mathfrak{g}_1, \mathfrak{g}_2$ are Lie algebras. Let (V_i, ρ_i) be a representation of $\mathfrak{g}_i$. Denote the following representation of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ by $(V_1, \rho_1) \boxtimes (V_2, \rho_2) := (V, \rho)$, where $V := V_1 \otimes V_2$ and $\rho(x, z) := \rho_1(x) \otimes \mathrm{id}_{V_2} + \mathrm{id}_{V_1} \otimes \rho_2(z)$.*

When dealing with compact Lie groups G_1 and G_2, it is known that a finite dimensional representation (W, ρ) of $G_1 \times G_2$ is irreducible if and only if $(W, \rho) \simeq (V_1, \rho_1) \boxtimes (V_2, \rho_2)$ for finite dimensional irreducible representations (V_i, ρ_i) of G_i [Sep07, Theorem 3.9].

Translating this to the language of Lie algebras, if $\mathfrak{g}_1, \mathfrak{g}_2$ are the Lie algebras of compact Lie groups, then a finite dimensional representation of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is irreducible if and only if it is isomorphic to $(V_1, \rho_1) \boxtimes (V_2, \rho_2)$ for finite dimensional irreducible representations (V_i, ρ_i) of $\mathfrak{g}_i$.

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The proof of this theorem uses character theory. However, character theory is not necessary for the case we are interested in: $\mathfrak{so}(4) \simeq \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. I am interested in this case because I use representations of $\mathfrak{so}(4)$ to study instantons with rotational symmetry (using the notation from my thesis).

Before we deal with our specific case, we first mention a useful property of $\boxtimes$.

Proposition 1.2. *Suppose that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Lie algebras. Let (V_i, ρ_i) and (W_i, λ_i) be representations of $\mathfrak{g}_i$. Then*

$$((V_1, \rho_1) \boxtimes (V_2, \rho_2)) \otimes ((W_1, \lambda_1) \boxtimes (W_2, \lambda_2)) \simeq ((V_1, \rho_1) \otimes (W_1, \lambda_1)) \boxtimes ((V_2, \rho_2) \otimes (W_2, \lambda_2)) \quad (1)$$

Proof. Let (V, ρ) denote the representation on the left-hand side of the statement and (W, λ) the representation on the right-hand side. Note that $V \simeq W$. We see that

$$\rho(x, z) := \rho_1(x) \otimes \mathrm{id}_{V_2} \otimes \mathrm{id}_{W_1} \otimes \mathrm{id}_{W_2} + \mathrm{id}_{V_1} \otimes \rho_2(z) \otimes \mathrm{id}_{W_1} \otimes \mathrm{id}_{W_2} + \mathrm{id}_{V_1} \otimes \mathrm{id}_{V_2} \otimes \lambda_1(x) \otimes \mathrm{id}_{W_2} + \mathrm{id}_{V_1} \otimes \mathrm{id}_{V_2} \otimes \mathrm{id}_{W_1} \otimes \lambda_2(z).$$

We can similarly write λ. Under the isomorphism $\phi\colon V \to W$, we see that $\lambda \circ \phi = \phi \circ \rho$. Thus, the representations are isomorphic. $\square$

2 Irreducible representations of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$

In this section, we classify all irreducible representations of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$.

Definition 2.1. *Let $v_1, \ldots, v_6 \in \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ denote the standard basis of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. That is, v_1, v_2, v_3 is the standard basis for $\mathfrak{sp}(1) \oplus 0$ and v_4, v_5, v_6 is the standard basis for $0 \oplus \mathfrak{sp}(1)$.*

Let (V, ρ) be an irreducible complex representation of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. Complexify the Lie algebra and, similar to the case of classifying representations of $\mathfrak{sp}(1)$, define

$$L^{\pm} := i\rho(v_1) \mp \rho(v_2), \quad L_3 := i\rho(v_3), \quad K^{\pm} := i\rho(v_4) \mp \rho(v_5), \quad K_3 := i\rho(v_6). \quad (2)$$

Then L_3 and K_3 span the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$.

Proposition 2.2. *The irreducible representations of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ are indexed by two integers. Given $m, n \in \mathbb{N}$, denote the irreducible representation with highest weight $\begin{bmatrix} \frac{m-1}{2} & \frac{n-1}{2} \end{bmatrix}$ by $(V_{m,n}, \rho_{m,n})$.*

Note 2.3. *The representation $(V_{m,n}, \rho_{m,n})$ is a mn-dimensional complex representation. Indeed, we generate mn distinct eigenvectors of L_3 and K_3 (when looking at the pair of eigenvalues together).*

Proof. Let $\lambda\colon \mathfrak{h} \to \mathbb{C}$ be the highest weight of the representation. Note that we can write λ as a row vector $\begin{bmatrix}\lambda_1 & \lambda_2\end{bmatrix}$ acting on $\mathfrak{h}$ by writing elements of $\mathfrak{h}$ in the L_3, K_3 basis and multiplying the matrices to get a complex number.

Note that $[L^\pm, L_3] = \mp L^\pm$ and $[K^\pm, K_3] = \mp K^\pm$. Additionally, $[L^+, L^-] = 2L_3$ and $[K^+, K^-] = 2K_3$. Finally, all other commutators vanish. Thus, we see that the roots of the Lie algebra are $\begin{bmatrix}\pm 1 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & \pm 1\end{bmatrix}$. We choose the positive roots to be $\begin{bmatrix}1 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 1\end{bmatrix}$. As λ is the highest weight of the representation, we have that the weight space $V_{\begin{bmatrix}\lambda_1 + m & \lambda_2 + n\end{bmatrix}} = 0$ for all $m, n \in \mathbb{Z}_{\geq 0}$, other than $m = n = 0$. Let $0 \neq v \in V_\lambda$. Hence, $L_3 v = \lambda_1 v$ and $K_3 v = \lambda_2 v$. Then we see that

$$L_3(L^+)^m(K^+)^n v = (\lambda_1 + m)(L^+)^m(K^+)^n v,$$
$$K_3(L^+)^m(K^+)^n v = (\lambda_2 + n)(L^+)^m(K^+)^n v.$$

Thus, $(L^+)^m(K^+)^n v \in V_{\begin{bmatrix}\lambda_1 + m & \lambda_2 + n\end{bmatrix}} = 0$. Hence, in particular, we have that $L^+ v = 0 = K^+ v$ (taking $(m, n) = (1, 0), (0, 1)$).

Note that $(K^-)^n v \in V_{\begin{bmatrix}\lambda_1 & \lambda_2 - n\end{bmatrix}}$ and $(L^-)^m v \in V_{\begin{bmatrix}\lambda_1 - m & \lambda_2\end{bmatrix}}$. As $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ is semisimple, the direct sum of the weight spaces gives the representation space. As the above weights are distinct, the eigenvectors are linearly independent. As we are dealing with finite dimensional representations, eventually there is some $m, n \in \mathbb{N}$ such that $(L^-)^{m-1}v, (K^-)^{n-1}v \neq 0$ but $(L^-)^m v = (K^-)^n v = 0$.

Using the commutation relations, one can show that

$$L^+(L^-)^m v = m(2\lambda_1 - m + 1)(L^-)^{m-1} v,$$
$$K^+(K^-)^n v = n(2\lambda_2 - n + 1)(K^-)^{n-1} v.$$

However, these must vanish, as $(K^-)^n v = 0$ and $(L^-)^m v = 0$. Hence, we see that $\lambda_1 = \frac{m-1}{2}$ and $\lambda_2 = \frac{n-1}{2}$. ∎

We show exactly what these irreducible representations look like (and also prove that they exist for all $m, n \in \mathbb{N}$).

The dimension of the centre of the universal enveloping algebra of a semi-simple Lie algebra equals the rank of the Lie algebra. Indeed, the Casimir operators are a basis for this space. In our case of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$, we have two Casimir operators.

Definition 2.4. *The* ***Casimir operators*** *for a representation* (V, ρ) *of* $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ *are*

$$C_1 := -\sum_{i=1}^{3} \rho(v_i)^2, \quad C_2 := -\sum_{i=4}^{6} \rho(v_i)^2. \tag{3}$$

Note that the sum of these Casimir operators gives the usual quadratic Casimir operator for this Lie algebra.

It is straightforward to see that these operators commute with $\rho(\upsilon, \omega)$ for all $\upsilon, \omega \in \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$, as they come from the Casimir operator for each summand of the direct sum.

Lemma 2.5. *Suppose we have an irreducible complex representation $(V_{m,n}, \rho_{m,n})$. As the Casimir operators commute with the representation, Schur's Lemma tells us that the Casimir operators are proportional to the identity. In particular, $C_1 = \frac{m^2-1}{4} I_{mn}$ and $C_2 = \frac{n^2-1}{4} I_{mn}$.*

Proof. Using the definitions of $L^{\pm}, K^{\pm}, L_3, K_3$, we see that $C_1 = L_3^2 + L_3 + L^- L^+$ and $C_2 = K_3^2 + K_3 + K^- K^+$.

Let $v \neq 0$ be a highest weight eigenvector. That is, $L_3 v = \frac{m-1}{2} v$ and $K_3 v = \frac{n-1}{2} v$. From above, we have that $L^+ v = 0 = K^+ v$. Using these identities, we see that $C_1 = \frac{m^2-1}{4} I_{mn}$ and $C_2 = \frac{n^2-1}{4} I_{mn}$. □

Therefore, we see that the Casimir operators classify the irreducible representations. Moreover, by examining their eigenvalues, they classify all representations.

Proposition 2.6. *The irreducible representation $(V_{m,n}, \rho_{m,n})$ exists for all $m, n \in \mathbb{N}$. Denote the irreducible, complex, a-dimensional representation of $\mathfrak{sp}(1)$ by (V_a, ρ_a). Then $(V_{m,n}, \rho_{m,n}) \simeq (V_m, \rho_m) \boxtimes (V_n, \rho_n)$.*

Proof. Consider the representation $(V_m, \rho_m) \boxtimes (V_n, \rho_n)$ of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. We see that the Casimir operators of this representation are $C_1 = \frac{m^2-1}{4} I_{mn}$ and $C_2 = \frac{n^2-1}{4} I_{mn}$. Therefore, $(V_m, \rho_m) \boxtimes (V_n, \rho_n) \simeq (V_{m,n}, \rho_{m,n})$.

In particular, such representations are irreducible and moreover, all irreducible representations of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ can be constructed in this way. □

Therefore, we have proven the original theorem in our case, without the use of character theory.

Note 2.7. *This approach does not work in general as the Casimir operators cannot generally be written in such a clear way, as they may not just be given by the quadratic Casimir operators of the summands of the direct sum. That is, being able to write C_1 and C_2 in terms of $L^{\pm}, K^{\pm}, L_3, K_3$ allowed us to prove this result.*

As $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \simeq \mathfrak{spin}(4)$ has Dynkin diagram D_2, its representations are self-dual. Therefore, its irreducible representations are either real or quaternionic. We prove this more directly.

Proposition 2.8. *When $m \equiv n \pmod 2$, we have that $(V_{m,n}, \rho_{m,n})$ is a real representation. Otherwise, $(V_{m,n}, \rho_{m,n})$ is a quaternionic representation.*

Note 2.9. *As all irreducible, complex representations are of real or quaternionic type, they are all self-dual. In particular, we have*

$$\begin{aligned}
\mathrm{Irr}(\mathfrak{spin}(4), \mathbb{C})_{\mathbb{R}} &= \{(V_{m,n}, \rho_{m,n}) \mid n \equiv m \pmod 2\},\\
\mathrm{Irr}(\mathfrak{spin}(4), \mathbb{C})_{\mathbb{C}} &= \emptyset,\\
\mathrm{Irr}(\mathfrak{spin}(4), \mathbb{C})_{\mathbb{H}} &= \{(V_{m,n}, \rho_{m,n}) \mid n \not\equiv m \pmod 2\}.
\end{aligned}$$

Thus, for $n \equiv m \pmod 2$, there is a unique irreducible real mn-representation $(\mathbb{R}^{mn}, \varrho_{m,n})$ whose complexification is $(V_{m,n}, \rho_{m,n})$. For $n \not\equiv m \pmod 2$, there is a unique irreducible real $2mn$-representation $(\mathbb{R}^{2mn}, \varrho_{m,n})$ whose complexification is $(V_{m,n}, \rho_{m,n})^{\oplus 2}$.

Moreover, when restricting the scalars of an irreducible quaternionic representation to $\mathbb{C}$, the complex representation is isomorphic to $(V_{m,n}, \rho_{m,n})$ for some $n \not\equiv m \pmod 2$ or $(V_{m,n}, \rho_{m,n})^{\oplus 2}$ for some $n \equiv m \pmod 2$.

Proof. Suppose that $m \equiv n \pmod 2$. We know that $(V_{m,n}, \rho_{m,n}) \simeq (V_m, \rho_m) \boxtimes (V_n, \rho_n)$. As (V_m, ρ_m) and (V_n, ρ_n) are either both real or both quaternionic representations of $\mathfrak{sp}(1)$, we have that there are conjugate-linear equivariant maps $J_m \colon V_m \to V_m$ and $J_n \colon V_n \to V_n$ such that $J_m^2 = J_n^2 = (\pm 1)\mathrm{id}$, where we have $+1$ if the m and n are odd and -1 if m and n are even.

Let $J_{m,n} := J_m \otimes J_n \colon V_{m,n} \to V_{m,n}$. We see that this map is conjugate-linear. Expanding the definition of $\rho_{m,n}$, we see that $J_{m,n}$ is equivariant. As $J_{m,n}^2 = (\pm 1)^2\mathrm{id} = \mathrm{id}$, we have that $(V_{m,n}, \rho_{m,n})$ is a real representation. That the other representations are quaternionic follows as all irreducible representations of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ must be one or the other. □

In order to understand tensor products, we need to know how to decompose them.

Proposition 2.10. *The tensor product of $(V_{a,b}, \rho_{a,b}) \otimes (V_{c,d}, \rho_{c,d})$ can be decomposed into a sum of irreducible representations as follows:*

$$(V_{a,b}, \rho_{a,b}) \otimes (V_{c,d}, \rho_{c,d}) \simeq \bigoplus_{i=1}^{\min(a,c)} \bigoplus_{j=1}^{\min(b,d)} (V_{a+c+1-2i, b+d+1-2j}, \rho_{a+c+1-2i, b+d+1-2j}). \quad (4)$$

Proof. Recall that $(V_{a,b}, \rho_{a,b})$ is given by $(V_a, \rho_a) \boxtimes (V_b, \rho_b)$. Then by Proposition 1.2,

$$(V_{a,b}, \rho_{a,b}) \otimes (V_{c,d}, \rho_{c,d}) \simeq ((V_a, \rho_a) \otimes (V_c, \rho_c)) \boxtimes ((V_b, \rho_b) \otimes (V_d, \rho_d)).$$

The result follows from the decomposition of tensor products of irreducible representations of $\mathfrak{sp}(1)$. □

References

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