Dual representations

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Abstract

We investigate the relationship between a finite-dimensional representation of a semi-simple Lie algebra and its dual. We show that for many semi-simple Lie algebras, all finite-dimensional representations are self-dual. That is, the dual of a representation is isomorphic to itself.

Keywords— representation theory

1 Identifying the Dual Representation

In this section, we show that the dual of an irreducible representation is irreducible and identify its highest weight. First, we define what the dual representation is.

Definition 1.1. *Suppose $\mathfrak{g}$ is a semi-simple Lie algebra with representation (V, ρ). The following representation (V^*, ρ^*) is known as the **dual representation**. Here $V^* := \mathrm{Hom}(V, \mathbb{C})$ and $\rho^*(x)f(v) := -f(\rho(x)v)$. A representation (V, ρ) is self-dual if $(V, \rho) \simeq (V^*, \rho^*)$.*

Note that we only look at finite-dimensional representations of semi-simple Lie algebras, so we assume all representations have finite dimension and all Lie algebras are semi-simple.

Proposition 1.2. *Suppose that (V, ρ) is irreducible. Then so too is (V^*, ρ^*).*

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Proof. Let $W \subseteq V^*$ be an invariant subspace. Let U be the linear subspace of V given by $U := \{v \in V \mid f(v) = 0, \forall f \in W\}$. Given $v \in U$, we see that for all $f \in W$,

$$f(\rho(x)v) = -\rho^*(x)f(v) = 0,$$

as $\rho^*(x)f \in W$, due to W being invariant. Thus, $\rho(x)v \in U$, so U is invariant.

Let $n := \dim_{\mathbb{C}}(V)$. As (V, ρ) is irreducible, $U = 0$ or $U = V$. If the latter is true, then let $f \in W$. For some $b_i \in \mathbb{C}$, we find $f = \sum_{i=1}^{n} b_i v_i^*$. But we know for all $v \in V$, $f(v) = 0$, so $f(v_i) = 0$ for all i. Hence, $f = 0$. That is, $W = 0$.

Suppose the former is true. Let $v_1 \in V \setminus \{0\}$. Then there exists $f_1 \in W$ such that $f_1(v_1) = 1$. Consider $\ker(f_1)$. This space is $(n-1)$-dimensional, so we choose $v_2 \in \ker(f_1) \setminus \{0\}$. Then there is some $f_2 \in W$ such that $f_2(v_2) = 1$. Suppose there are some $c_1, c_2 \in \mathbb{C}$ such that $c_1 f_1 + c_2 f_2 = 0$. Evaluating at v_2 implies $c_2 = 0$, so $c_1 = 0$. That is, f_1 and f_2 are linearly independent. Consider $\ker(f_1) \cap \ker(f_2)$. This space is $(n-2)$-dimensional, so we can choose $v_3 \in (\ker(f_1) \cap \ker(f_2)) \setminus \{0\}$. There exists $f_3 \in W$ such that $f_3(v_3) = 1$. We can repeat until we get a linearly independent set of $f_1, \dots, f_n \in W$. Hence, $W = V^*$.

Thus, the only invariant subspaces of V^* are 0 and V^* itself, so we see that (V^*, ρ^*) is irreducible. □

Proposition 1.3. *Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$, a semi-simple Lie algebra, and $\lambda \in \mathfrak{h}^*$. Then λ is a weight of (V, ρ) if and only if $-\lambda$ is a weight of (V^*, ρ^*), with the same multiplicity.*

Proof. Suppose λ is a weight of (V, ρ). The weight space of a weight $\lambda \colon \mathfrak{h} \to \mathbb{C}$ is given by

$$V_\lambda := \{v \in V \mid \forall H \in \mathfrak{h}, \rho(H)v = \lambda(H)v\}.$$

Consider $(V^*)_{-\lambda} = \{f \in V^* \mid \forall H \in \mathfrak{h}, \rho^*(H)f = -\lambda(H)f\}$. Let $H \in \mathfrak{h}$ and $\{v_1, \dots, v_m\} \subseteq V_\lambda$ a basis. As $\mathfrak{g}$ is semi-simple, we have that V decomposes into a direct sum of its weight spaces. That is, $V \simeq \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$. Extend the basis $\{v_1, \dots, v_m\}$ of V_λ to a basis $\{v_1, \dots, v_m, w_1, \dots, w_{n-m}\} \subseteq V$ where the w_i belong to different weight spaces.

Define $f_i \colon \mathbb{V} \to \mathbb{C}$ by the linear map taking $v_i \mapsto 1$, $v_j \mapsto 0$, for all $j \neq i$ and $w_k \mapsto 0$, for all k. Thus, we get a linearly independent set $\{f_1, \dots, f_n\} \in V^*$. We show that each $f_i \in (V^*)_{-\lambda}$.

Indeed, let $v \in V$. We can write $v = \sum_{j=1}^{m} \alpha_j v_j + \sum_{k=1}^{n-m} \beta_k w_k$. Then

$$\rho^*(H)f_i(v) = -f_i\left(\sum_{j=1}^{m} \alpha_j \rho(H) v_j + \sum_{k=1}^{n-m} \beta_k \rho(H) w_k\right).$$

As the $v_j \in V_\lambda$, we have $\rho(H)v_j = \lambda(H)v_j$. For the w_k, there is some weight $\mu_k \in \mathfrak{h}^*$ such that $\rho(H)w_k = \mu_k(H)w_k$. Then, we see

$$\rho^*(H)f_i(v) = -\lambda(H)f_i\left(\sum_{j=1}^{m} \alpha_j v_j\right) = -\lambda(H)f_i(v).$$

Therefore, $(V^*)_{-\lambda}$ has the same dimension as V_λ.

Conversely, suppose that $-\lambda$ is a weight of (V^*, ρ^*). As $((V^*)^*, (\rho^*)^*) \simeq (V, \rho)$, we have that λ is a weight of (V, ρ) with the same multiplicity. □

Now that we understand the weights of the dual representation, we need only identify the highest weight.

Definition 1.4. *Given a root system Φ in E, a finite dimensional Euclidean vector space, the* ***Weyl group*** *W is the subgroup of the isometry group generated by reflections through the hyperplanes orthogonal to the roots. Hence, it is a finite reflection group. It turns out that this group is generated by reflections through the hyperplanes orthogonal to the simple roots, so we need only consider these roots.*

Let $\langle \cdot, \cdot \rangle$ be the inner product on E. The complement of the set of hyperplanes above is disconnected. Each connected component is called a ***Weyl chamber****. If we have a set of simple roots Δ, then the* ***fundamental Weyl chamber*** *is the set of points $v \in E$ such that $(\alpha, v) > 0$ for all $\alpha \in \Delta$.*

There is a well-known theorem relating the Weyl group and Weyl chambers.

Theorem 1.5. *The Weyl group acts freely and transitively on the Weyl chambers.*

Definition 1.6. *Based on the above, there is a unique $w_0 \in W$ mapping the fundamental Weyl chamber to its negative chamber and vice versa. Such a map sends $\Delta \mapsto -\Delta$ and vice versa. This element is called the* ***longest element*** *of the Weyl group.*

Proposition 1.7. *Let (V, ρ) be an irreducible (complex) representation of $\mathfrak{g}$, with highest weight λ. Then $w_0(\lambda)$ is the lowest weight of V.*

Proof. Let $\{\alpha_1, \dots, \alpha_m\} = \Delta$. Note that if μ is a weight, then so too is $w_0(\mu)$, as an element of the Weyl group permutes the set of weights. Thus, we have for a weight μ, as λ is the highest weight, there exists $a_i \in \mathbb{Z}_{\geq 0}$ such that

$$\lambda - w_0(\mu) = \sum_{i=1}^{m} a_i \alpha_i.$$

Note that as w_0 is a reflection, $w_0^2 = \mathrm{id}_E$. We see that

$$w_0(\lambda) - \mu = \sum_{i=1}^{m} a_i w_0(\alpha_i).$$

Note that as $\alpha_i \in \Delta$, we have $w_0(\alpha_i) \in -\Delta$. Hence, $-w_0(\alpha_i) \in \Delta$. As w_0 acts freely on the set of simple roots, we have that $\{-w_0(\alpha_1), -\dots, -w_0(\alpha_m)\} = \Delta$. Therefore, we see that

$$w_0(\lambda) - \mu = \sum_{i=1}^{m} (-a_i)(-w_0(\alpha_i)).$$

As all $-a_i \leq 0$, we see that $w_0(\lambda)$ is the lowest weight. □

Corollary 1.8. *Let λ be the highest weight of an irreducible representation (V, ρ). Then $-w_0(\lambda)$ is the highest weight of (V^*, ρ^*).*

Proof. Let $-\mu$ be a weight of (V^*, ρ^*). By Proposition 1.3, μ is a weight of (V, ρ). By Proposition 1.7, we see that

$$-w_0(\lambda) + \mu = -\sum_{i=1}^{m} (-a_i)(-w_0(\alpha_i)) = \sum_{i=1}^{m} a_i(-w_0(\alpha_i)).$$

As $\{-w_0(\alpha_i)\} = R^+$ and $a_i \geq 0$, we see that $-w_0(\lambda)$ is the highest weight of the dual representation (V^*, ρ^*). □

2 Lie algebras with all self-dual representations

In this section, we compute the map w_0 for all semi-simple Lie algebras. First, we must note that an irreducible representation is determined by its highest weight $\sum_{i=1}^{m} a_i \alpha_i$, where $a_i \geq 0$ and α_i are simple roots. On the Dynkin diagram associated to the Lie algebra, simple roots correspond with the nodes. Thus, we can find a correspondence between irreducible Lie algebra representations and decorated Dynkin diagrams, where we adorn the Dynkin diagram with the a_i appearing above the corresponding node.

The element $-w_0 \colon \Delta \to \Delta$ is an isometry, so it provides an automorphism on the Dynkin diagram. If this automorphism is the identity, then every representation is self-dual. If not, then we can see what the dual representation is, by applying this automorphism to the decorated Dynkin diagrams. Note that this means that for $B_n, C_n, E_7, E_8, F_4, G_2$, $-w_0 = \mathrm{id}$, as there are no nontrivial diagram automorhpisms. Nevertheless, we show this more concretely below.

Definition 2.1. *Let α_i be a simple root, with **coroot** $\alpha_i^\vee := \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$. The **reflection about the root** α_i is given by*

$$r_i(x) := x - \langle \alpha_i^\vee, x \rangle \alpha_i. \tag{1}$$

*The **fundamental roots** are the elements ϖ_i defined such that $\langle \alpha_i^\vee, \varpi_j \rangle = \delta_{ij}$. Note that*

$$r_i(\varpi_j) = \begin{cases} \varpi_j & i \neq j \\ \varpi_j - \alpha_j & i = j \end{cases}. \tag{2}$$

We use the root systems given by Humphreys [Hum72, Chapter 12]. Below, let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbb{R}^m$ and let $\langle \cdot, \cdot \rangle$ be the usual inner product.

Proposition 2.2. *For $A_1 = B_1 = C_1$, we have $-w_0 = \mathrm{id}$, so all representations are self-dual. For A_n, $n \geq 2$, we have that $-w_0 \neq \mathrm{id}$. In either case, the diagram automorphism flips the diagram horizontally (for A_1 this automorphism is the identity).*

Proof. Labelling the roots left to right as $\alpha_1, \ldots, \alpha_n$, we define $\alpha_i := e_i - e_{i+1}$. Then we see that

$$r_i(e_j) = \begin{cases} e_j & j \neq i, i+1 \\ e_{i+1} & j = i \\ e_i & j = i+1 \end{cases}. \tag{3}$$

As these reflections generate the Weyl group W, we see that $W \simeq S_{n+1}$, the permutation group on $n+1$ elements. Consider the permutation sending $e_i \mapsto e_{n+2-i}$ for $i = 1, \ldots, n+1$. Such a permutation sends all $\alpha_i \mapsto -\alpha_{n+1-i}$. Hence, it takes $\Delta \mapsto -\Delta$ and vice versa. However, only one element of the Weyl group can do this, w_0.

Therefore, for $n = 1$, $-w_0 = \mathrm{id}$, as $-w_0(\alpha_1) = \alpha_1$. Thus, for A_1, all representations are self-dual. However, for $n > 1$, we see that $-w_0 \neq \mathrm{id}$ flips the diagram horizontally. □

Proposition 2.3. *For B_n, $n \geq 2$, we have $-w_0 = \mathrm{id}$, so all representations are self-dual. That is, the diagram automorphism is the identity (as expected).*

Proof. Labelling the roots left to right as $\alpha_1, \ldots, \alpha_n$, define $\alpha_i := e_i - e_{i+1}$ for $i \in \{1, \ldots, n-1\}$ and $\alpha_n := e_n$ (the short root). Then we see that for $i \leq n-1$, we have

$$r_i(e_j) = \begin{cases} e_j & j \neq i, i+1 \\ e_{i+1} & j = i \\ e_i & j = i+1 \end{cases}. \tag{4}$$

Additionally, we have

$$r_n(e_j) = \begin{cases} e_j & j \neq n \\ -e_n & j = n \end{cases} . \tag{5}$$

Thus, W can permute any e_j and negate any number of them. Take the element sending $e_i \mapsto -e_i$ for all $i \in \{1, \ldots, n\}$. This element sends all $\alpha_i \mapsto -\alpha_i$, hence taking $\Delta \mapsto -\Delta$ and vice versa. Therefore, this element is w_0 and we see $-w_0 = \text{id}$. ☐

Proposition 2.4. *For C_n, $n \geq 2$, we have $-w_0 = \text{id}$, so all representations are self-dual. That is, the diagram automorphism is the identity (as expected).*

Proof. Labelling the roots left to right as $\alpha_1, \ldots, \alpha_n$, define $\alpha_i := e_i - e_{i+1}$ for $i \in \{1, \ldots, n-1\}$ and $\alpha_n := 2e_n$ (the long root). Then we see that for $i \leq n-1$, we have

$$r_i(e_j) = \begin{cases} e_j & j \neq i, i+1 \\ e_{i+1} & j = i \\ e_i & j = i+1 \end{cases} . \tag{6}$$

Additionally, we have

$$r_n(e_j) = \begin{cases} e_j & j \neq n \\ -e_n & j = n \end{cases} . \tag{7}$$

Thus, W can permute any e_j and negate any number of them. Take the element sending $e_i \mapsto -e_i$ for all $i \in \{1, \ldots, n\}$. This element sends all $\alpha_i \mapsto -\alpha_i$, hence taking $\Delta \mapsto -\Delta$ and vice versa. Therefore, this element is w_0 and we see $-w_0 = \text{id}$. ☐

Proposition 2.5. *For D_n, $n \geq 2$, we have $-w_0 = \text{id}$ if n is even, so all representations are self-dual. That is, the diagram automorphism is the identity. If n is odd, we have $-w_0 \neq \text{id}$. The diagram automorphism flips the diagram vertically.*

Proof. Labelling the roots left to right as $\alpha_1, \ldots, \alpha_{n-2}$, then α_{n-1} the top root and α_n the bottom root, define $\alpha_i := e_i - e_{i+1}$ for $i \in \{1, \ldots, n-1\}$ and $\alpha_n := e_{n-1} + e_n$. Then we see that for $i \leq n-1$, we have

$$r_i(e_j) = \begin{cases} e_j & j \neq i, i+1 \\ e_{i+1} & j = i \\ e_i & j = i+1 \end{cases} . \tag{8}$$

Additionally, we have

$$r_n(e_j) = \begin{cases} e_j & j \neq n-1, n \\ -e_n & j = n-1 \\ -e_{n-1} & j = n \end{cases} . \tag{9}$$

Thus, W can permute any e_j and negate any even number of them.

If n is even, take the element sending $e_i \mapsto -e_i$ for all $i \in \{1, \dots, n\}$. This element sends all $\alpha_i \mapsto -\alpha_i$, hence taking $\Delta \mapsto -\Delta$ and vice versa. Therefore, this element is w_0 and we see $-w_0 = \mathrm{id}$.

If n is odd, take the element sending $e_i \mapsto -e_i$ for $i \in \{1, \dots, n-1\}$ and $e_n \mapsto e_n$. Then for all $i \in \{1, \dots, n-2\}$ we have $\alpha_i \mapsto -\alpha_i$. We also have $\alpha_{n-1} \mapsto -\alpha_n$ and $\alpha_n \mapsto -\alpha_{n-1}$. Thus, $\Delta \mapsto -\Delta$ and vice versa, so this element is w_0. Therefore, $-w_0 \neq \mathrm{id}$ and we see the diagram automorphism flips the diagram vertically. □

Proposition 2.6. *For F_4, we have $-w_0 = \mathrm{id}$, so all representations are self-dual. That is, the diagram automorphism is the identity (as expected).*

Proof. Labelling the roots left to right as $\alpha_1, \dots, \alpha_4$, define $\alpha_1 := e_2 - e_3$, $\alpha_2 := e_3 - e_4$, $\alpha_3 := e_4$, and $\alpha_4 := \frac{e_1 - e_2 - e_3 - e_4}{2}$ (where α_3, α_4 are the short roots). One can compute that the following Weyl element, in the $\{e_1, e_2, e_3, e_4\}$ basis, is given by

$$(r_1 \circ r_3 \circ r_2 \circ r_4)^6 = -I_4. \tag{10}$$

Thus, this element takes $\alpha_i \mapsto -\alpha_i$ for $i = 1, 2, 3, 4$, so it takes $\Delta \mapsto -\Delta$ and vice versa. Hence, this element is w_0 and $-w_0 = \mathrm{id}$. □

Proposition 2.7. *For G_2, we have $-w_0 = \mathrm{id}$, so all representations are self-dual. That is, the diagram automorphism is the identity (as expected).*

Proof. Labelling the roots left to right as α_1 and α_2, define $\alpha_1 := e_1 - e_2$ and $\alpha_2 := -2e_1 + e_2 + e_3$ (the long root). One can compute that the following Weyl element, in the $\{e_1, e_2, e_3\}$ basis, is given by

$$(r_1 \circ r_2)^3 = \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}. \tag{11}$$

Thus, this element takes $\alpha_i \mapsto -\alpha_i$ for $i = 1, 2$, so it takes $\Delta \mapsto -\Delta$ and vice versa. Hence, this element is w_0 and $-w_0 = \mathrm{id}$. □

Proposition 2.8. *For E_7, E_8, we have $-w_0 = \mathrm{id}$, so all representations are self-dual. That is, the diagram automorphism is the identity (as expected). For E_6, we have $-w_0 \neq \mathrm{id}$. The diagram automorphism in this case flips the diagram horizontally.*

Proof. For E_n, labelling the roots left to right as $\alpha_1, \dots, \alpha_{n-1}$ and α_n as the remaining root not in the line, define $\alpha_1 := \frac{1}{2}\left(e_1 + e_8 - \sum_{i=2}^{7} e_i\right)$, $\alpha_i := e_i - e_{i-1}$ for $i = 2, \dots, n-1$, and $\alpha_n := e_1 + e_2$.

Consider $n = 8$. One can compute the following Weyl element, in the $\{e_1, \dots, e_8\}$ basis, is given by

$$(r_1 \circ r_3 \circ r_5 \circ r_7 \circ r_2 \circ r_4 \circ r_6 \circ r_8)^{15} = -I_8. \tag{12}$$

Thus, this element takes $\alpha_i \mapsto -\alpha_i$ for $i = 1, \dots, 8$, so it takes $\Delta \mapsto -\Delta$ and vice versa. Hence, this element is w_0 and $-w_0 = \mathrm{id}$.

Consider $n = 7$. One can compute the following Weyl element, in the $\{e_1, \dots, e_8\}$ basis, is given by

$$(r_1 \circ r_3 \circ r_5 \circ r_2 \circ r_4 \circ r_6 \circ r_7)^9 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{13}$$

Thus, this element takes $\alpha_i \mapsto -\alpha_i$ for $i = 1, \dots, 7$, so it takes $\Delta \mapsto -\Delta$ and vice versa. Hence, this element is w_0 and $-w_0 = \mathrm{id}$.

Consider $n = 6$. One can compute the following Weyl element, in the $\{e_1, \dots, e_8\}$ basis, is given by

$$(r_1 \circ r_3 \circ r_5 \circ r_2 \circ r_4 \circ r_6)^6 = \frac{1}{2} \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix}, \tag{14}$$

where

$$M := \begin{bmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix}. \tag{15}$$

Thus, this element takes $\alpha_i \mapsto -\alpha_{6-i}$ for $i = 1, \dots, 5$ and $\alpha_6 \mapsto -\alpha_6$ so it takes $\Delta \mapsto -\Delta$ and vice versa. Hence, this element is w_0 and $-w_0 \neq \mathrm{id}$. In particular, we see the diagram automorphism flips the diagram horizontally. $\square$

References

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