# Real and quaternionic representations

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**Abstract**

Following Bröcker and Dieck, adapting results for compact Lie groups to Lie algebra, we investigate what it means to be a real, complex, or quaternionic representation and the relationships between them [BTD85, Ch. II, § 6]. We also classify the representations of $\mathfrak{sp}(1)$.

***Keywords*** — representation theory

## 1 Different types of representations

In this section, we define what it means to be a real, complex, or quaternionic representation as well as functors between them. We then examine the relationships between these functors.

**Definition 1.1.** *A* ***(complex) representation*** *of a Lie algebra $\mathfrak{g}$ is a pair $(V, \rho)$, where $V$ is a complex vector space and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra homomorphism. That is, $\rho$ is linear and for all $x, y \in \mathfrak{g}$, $\rho$ satisfies $\rho([x, y]) = [\rho(x), \rho(y)]$. If $V$ is a real (resp. quaternionic) vector space and $\rho$ is $\mathbb{R}$-linear (resp. $\mathbb{H}$-linear), then $(V, \rho)$ is called a* ***real (resp. quaternionic) representation****.*

**Definition 1.2.** *Let $(V, \rho)$ be a representation of $\mathfrak{g}$. We say that an inner product $\langle \cdot, \cdot \rangle$ on $V$ is $\mathfrak{g}$-invariant if*

$$\langle \rho(x)v, w \rangle + \langle v, \rho(x)w \rangle = 0.$$

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**Proposition 1.3.** *If $\mathfrak{g}$ is the Lie algebra of a compact Lie group and $(V, \rho)$ a representation of $\mathfrak{g}$, then there is a $\mathfrak{g}$-invariant inner product on $V$.*

*Proof.* This result is true because all representations of compact Lie groups have a $G$-invariant inner product on $V$. Thus, we consider the associated Lie algebra representation and find that the inner product is $\mathfrak{g}$-invariant. However, we need the Lie group to be simply-connected for all representations of $\mathfrak{g}$ to arise this way. If $\mathfrak{g}$ is the Lie algebra of a compact Lie group, then it is the Lie algebra of the connected component of this Lie group containing the identity. Furthermore, it is the Lie algebra of the universal cover of this connected, compact Lie group, which is a compact, simply-connected Lie group. □

Based on the above result, we only consider Lie algebras that are the Lie algebra of a compact Lie group.

**Definition 1.4.** *As we always have a $\mathfrak{g}$-invariant inner product, we consider* ***unitary***, ***orthogonal***, *and* ***symplectic*** *representations $(V, \rho)$, where the image of $\rho$ is contained in $\mathfrak{su}(k)$, $\mathfrak{so}(k)$, and $\mathfrak{sp}(k)$, respectively. For such Lie algebras, one can show that every complex, real, and quaternionic representation is a direct sum of irreducible representations; such representations are semi-simple.*

**Definition 1.5.** *Let $(V, \rho)$ be a complex representation. A* ***real (resp. quaternionic) structure*** *on $(V, \rho)$ is a conjugate-linear equivariant map $J \colon V \to V$ such that $J^2 = \mathrm{id}_V$ (resp. $J^2 = -\mathrm{id}_V$). Explicitly, a conjugate-linear equivariant map is one where for all $v, w \in V$, $\alpha \in \mathbb{C}$, and $x \in \mathfrak{g}$, we have*

$$J(v + w) = J(v) + J(w), \quad J(\alpha v) = \overline{\alpha} J(v), \quad J(\rho(x)v) = \rho(x)J(v).$$

*A complex representation $(V, \rho)$ is said to be of* ***real (resp. quaternionic) type*** *if it admits a real (resp. quaternionic) structure.*

Note that a representation can be both quaternionic and real type. For example, consider $(\mathbb{H}, 0)$. Using $\mathbb{H} \simeq \mathbb{C}^2$, we take $v \mapsto \overline{v}$ and $v \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \overline{v}$ to get a real and quaternionic structure, respectively.

We define some categories of representations and functors between them.

**Definition 1.6.** *Let $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and let $\mathrm{Rep}(\mathfrak{g}, K)$ be the category whose objects are representations on $K$ vector spaces and whose morphisms are $K$-linear $\mathfrak{g}$ equivariant maps.*

*Let $\mathrm{Rep}_+(\mathfrak{g}, \mathbb{C})$ and $\mathrm{Rep}_-(\mathfrak{g}, \mathbb{C})$ be the categories of complex representations with real and quaternionic structures, respectively. Morphisms in these categories are $\mathbb{C}$-linear equivariant maps which commute with the structure maps.*

**Proposition 1.7.** *The categories* $\mathrm{Rep}(\mathfrak{g},\mathbb{R})$ *and* $\mathrm{Rep}_+(\mathfrak{g},\mathbb{C})$ *are equivalent. That is, real representations are just complex representations with additional structures. Similarly,* $\mathrm{Rep}(\mathfrak{g},\mathbb{H})$ *and* $\mathrm{Rep}_-(\mathfrak{g},\mathbb{C})$ *are equivalent categories.*

*Proof.* We construct functors $e_+\colon \mathrm{Rep}(\mathfrak{g},\mathbb{R}) \to \mathrm{Rep}_+(\mathfrak{g},\mathbb{C})$ and $s_+\colon \mathrm{Rep}_+(\mathfrak{g},\mathbb{C}) \to \mathrm{Rep}(\mathfrak{g},\mathbb{R})$ as follows. Given a real representation $(U,\rho)$, let $e_+(U,\rho) := (\mathbb{C}\otimes_{\mathbb{R}} U, \mathrm{id}_{\mathbb{C}}\otimes \rho, J)$ with structure map $J(\alpha\otimes u) := \overline{\alpha}\otimes u$. Given a representation $(V,\rho,J)$, let $V_\pm$ be the $\pm 1$-eigenspaces of $J$. As $2v = (v+J(v)) + (v-J(v))$, we have $V = V_+ \oplus V_-$. Note that multiplication by $i$ induces $V_+ \simeq V_-$. Let $s_+(V,\rho,J) := (V_+,\rho)$, where $\rho$ is restricted to act on $V_+$. Note that $\rho(x)\mid_{V_+}\colon V_+ \to V_+$. Indeed, if $v\in V_+$, then $J(v) = v$, so

$$\rho(x)v = \rho(x)J(v) = J(\rho(x)v),$$

so $\rho(x)v \in V_+$.

The compositions $e_+\circ s_+$ and $s_+\circ e_+$ are naturally equivalent to the identity. Indeed,

$$e_+ \circ s_+(V,\rho,\tilde{J}) = e_+(V_+,\rho) = (\mathbb{C}\otimes_{\mathbb{R}} V_+, \mathrm{id}_{\mathbb{C}}\otimes\rho, J).$$

But this representation is equivalent to $(V,\rho,\tilde{J})$ via the linear isomorphism $\phi\colon \mathbb{C}\otimes_{\mathbb{R}} V_+ \to V$ taking $\phi(\alpha\otimes v) := \alpha v$. Note that

$$\begin{aligned}\phi((\mathrm{id}_{\mathbb{C}}\otimes\rho)(x)(\alpha\otimes v)) = \phi(\alpha\otimes\rho(x)v) = \alpha\rho(x)v = \rho(x)\phi(\alpha\otimes v);\\ \phi(J(\alpha\otimes v)) = \phi(\overline{\alpha}\otimes v) = \overline{\alpha}v = \tilde{J}(\alpha v) = \tilde{J}(\phi(\alpha\otimes v)).\end{aligned}$$

Above we use that $v\in V_+$. Similarly,

$$s_+\circ e_+(U,\rho) = s_+(\mathbb{C}\otimes_{\mathbb{R}} U, \mathrm{id}_{\mathbb{C}}\otimes\rho, J) = ((\mathbb{C}\otimes_{\mathbb{R}} U)_+, \mathrm{id}_{\mathbb{C}}\otimes\rho).$$

As $J(\alpha\otimes u) = \overline{\alpha}\otimes u$, $(\mathbb{C}\otimes_{\mathbb{R}} U)_+ = \{1\otimes u \mid u\in U\}$. This representation is equivalent to $(U,\rho)$ via the linear isomorphism $\psi\colon (\mathbb{C}\otimes_{\mathbb{R}} U)_+ \to U$ taking $\psi(1\otimes u) := u$. Note that

$$\psi((\mathrm{id}_{\mathbb{C}}\otimes\rho)(x)(1\otimes u)) = \psi(1\otimes\rho(x)u) = \rho(x)u = \rho(x)\psi(1\otimes u).$$

We construct functors $e_-\colon \mathrm{Rep}(\mathfrak{g},\mathbb{H}) \to \mathrm{Rep}_-(\mathfrak{g},\mathbb{C})$ and $s_-\colon \mathrm{Rep}_-(\mathfrak{g},\mathbb{C}) \to \mathrm{Rep}(\mathfrak{g},\mathbb{H})$ as follows. Let $(W,\rho)$ be a quaternionic representation. Choose a $\mathbb{H}$-basis $\{e_1,\dots,e_n\}$ of $W$. Let $V_1 := \mathrm{span}_{\mathbb{C}}(e_1,\dots,e_n)$ and $V_2 := \mathrm{span}_{\mathbb{C}}(je_1,\dots,je_n)$. As a complex vector space, $W$ is isomorphic to $V_1\oplus V_2$. Let $\rho_{1ij},\rho_{2ij}\in\mathbb{C}$ such that $\rho(x)e_i = \rho_{1ij}e_j + \rho_{2ij}je_j$. As $\rho$ is $\mathbb{H}$-linear, we have that $\rho(x)je_i = j\rho(x)e_i = \overline{\rho_{1ij}}je_j - \overline{\rho_{2ij}}e_j$. Written in the $\mathbb{C}$-basis, $\rho(x)$ acts on $V_1\oplus V_2$ as $\tilde{\rho}(x) := \begin{bmatrix}\rho_1 & -\overline{\rho_2}\\ \rho_2 & \overline{\rho_1}\end{bmatrix}$. Moreover, we can define $J$ to be the action of $j$ on $V_1\oplus V_2$. Explicitly, $J\colon V_1\oplus V_2\to V_1\oplus V_2$ takes

$J\left(\alpha_i e_i, \beta_j j e_j\right) := (-\overline{\beta_j} e_j, \overline{\alpha_i} j e_i)$. Note that $J$ is conjugate-linear and equivariant, satisfying $J^2 = -\mathrm{id}_{V_1 \oplus V_2}$. Indeed, we see

$$\begin{aligned} J(\tilde{\rho}(x)(e_i, 0)) &= J(\rho_{1ij} e_j, \rho_{2ij} j e_j) = (-\overline{\rho_{2ij}} e_j, \overline{\rho_{1ij}} j e_j) = \tilde{\rho}(x) J(e_i, 0), \\ J(\tilde{\rho}(x)(0, j e_i)) &= J(-\overline{\rho_{2ij}} e_j, \overline{\rho_{1ij}} j e_j) = (-\rho_{1ij} e_i, -\rho_{2ij} j e_j) = \tilde{\rho}(x) J(0, j e_i). \end{aligned}$$

Then $e_-(W, \rho) := (V_1 \oplus V_2, \tilde{\rho}, J)$. Given a representation $(V, \rho, J)$, define $f\colon \mathbb{H} \times V \to V$ as follows. Given $p \in \mathbb{H}$, we can write it uniquely as $p = \alpha + \beta j$ for $\alpha, \beta \in \mathbb{C}$. Then let $p \cdot v \equiv f(p, v) := \alpha v + \beta J(v)$. Because $J$ is conjugate-linear and $J^2 = -\mathrm{id}_V$, we have that this gives $V$ the structure of a quaternionic vector space. Note that using this quaternionic scalar multiplication, we have that as $J$ is equivariant and $\rho(x)$ is $\mathbb{C}$-linear,

$$\rho(x) p \cdot v = \rho(x)(\alpha v + \beta J(v)) = \alpha \rho(x) v + \beta J(\rho(x) v) = p \cdot \rho(x) v.$$

Thus, $\rho(x)$ is $\mathbb{H}$-linear. Then let $s_-(V, \rho, J) := (V, \rho)$, where we view $V$ as a quaternionic vector space and $\rho(x)$ as $\mathbb{H}$-linear.

The compositions $e_- \circ s_-$ and $s_- \circ e_-$ are naturally equivalent to the identity. Indeed,

$$s_- \circ e_-(W, \rho) = s_-(V_1 \oplus V_2, \tilde{\rho}, J) = (V_1 \oplus V_2, \tilde{\rho}).$$

This representation is equivalent to $(W, \rho)$ via the $\mathbb{H}$-linear isomorphism $\phi\colon W \to V_1 \oplus V_2$ sending basis vectors to $\phi(e_i) := (e_i, 0)$. Note that $\phi(j e_i) = j \cdot \phi(e_i) = J(e_i, 0) = (0, e_i)$. Hence, $\phi$ is indeed bijective and

$$\phi(p e_i) = p \cdot \phi(e_i) = (\alpha e_i, \beta e_i).$$

We see that

$$\tilde{\rho}(x) \phi(e_i) = \begin{bmatrix} \rho_1 & -\overline{\rho_2} \\ \rho_2 & \overline{\rho_1} \end{bmatrix} \begin{bmatrix} e_i \\ 0 \end{bmatrix} = (\rho_{1ij} e_j, \rho_{2ij} j e_j) = \phi(\rho(x) e_i).$$

Similarly, we have

$$e_- \circ s_-(V, \rho, \tilde{J}) = e_-(V, \rho) = (V_1 \oplus V_2, \tilde{\rho}, J).$$

This representation is equivalent to $(V, \rho, \tilde{J})$ via the $\mathbb{C}$-linear isomorphism $\phi\colon V \to V_1 \oplus V_2$ given as follows. Let $\{e_1, \ldots, e_n\}$ be a $\mathbb{H}$-basis for $V$. Then $\{e_1, \ldots, e_n\} \cup \{\tilde{J}(e_1), \ldots, \tilde{J}(e_n)\}$ is a $\mathbb{C}$-basis for $V$. Let $\phi$ send basis vectors to $\phi(e_i) := (e_i, 0)$ and $\phi(\tilde{J}(e_i)) := (0, j \cdot e_i)$. We see that $\phi$ is bijective. Recall that $\rho_{1ij}$ and $\rho_{2ij}$ were defined such that $\rho(x) e_i = \rho_{1ij} e_j + \rho_{2ij} j \cdot e_j$. Note that $j \cdot e_j = \tilde{J}(e_j)$. Then we see

$$\begin{aligned} \tilde{\rho}(x) \phi(e_i) &= (\rho_{1ij} e_j, \rho_{2ij} j \cdot e_j) = \phi(\rho_{1ij} e_j + \rho_{2ij} \tilde{J}(e_j)) = \phi(\rho(x) e_i), \\ \tilde{\rho}(x) \phi(\tilde{J}(e_i)) &= (-\overline{\rho_{2ij}} e_j, \overline{\rho_{1ij}} j \cdot e_j) = \phi(\tilde{J}(\rho(x) e_i)) = \phi(\rho(x) \tilde{J}(e_i)), \\ J(\phi(e_i)) &= J(e_i, 0) = (0, j \cdot e_i) = \phi(\tilde{J}(e_i)), \\ J(\phi(\tilde{J}(e_i))) &= J(0, j \cdot e_i) = (-e_i, 0) = \phi(\tilde{J}(\tilde{J}(e_i))). \end{aligned}$$ □

**Note 1.8.** *If $(V, \rho, J)$ is a complex representation of quaternionic type, then $\dim_{\mathbb{C}}(V)$ is even. In Proposition 1.7, we see that $V$ has the structure of a $\mathbb{H}$-vector space. Thus, we can find a $\mathbb{H}$-basis $\{e_1, \dots, e_n\}$, which gives us a $\mathbb{C}$-basis $\{e_1, \dots, e_n, je_1, \dots, je_n\}$. Thus, $\dim_{\mathbb{C}}(V) = 2n$.*

We can also define functors between different types of representations.

**Definition 1.9.** *Forgetting about various parts of the representations, we have restriction maps*

$$\begin{aligned}
&r_{\mathbb{R}}^{\mathbb{C}} \colon \mathrm{Rep}(\mathfrak{g}, \mathbb{C}) \to \mathrm{Rep}(\mathfrak{g}, \mathbb{R}),\\
&r_{\mathbb{C}}^{\mathbb{H}} \colon \mathrm{Rep}(\mathfrak{g}, \mathbb{H}) \to \mathrm{Rep}(\mathfrak{g}, \mathbb{C}),\\
&r_{\mathbb{R}}^{\mathbb{H}} \colon \mathrm{Rep}(\mathfrak{g}, \mathbb{H}) \to \mathrm{Rep}(\mathfrak{g}, \mathbb{R}), \quad r_{\mathbb{R}}^{\mathbb{H}} = r_{\mathbb{R}}^{\mathbb{C}} \circ r_{\mathbb{C}}^{\mathbb{H}},\\
&r_{+} \colon \mathrm{Rep}_{+}(\mathfrak{g}, \mathbb{C}) \to \mathrm{Rep}(\mathfrak{g}, \mathbb{C}),\\
&r_{-} \colon \mathrm{Rep}_{-}(\mathfrak{g}, \mathbb{C}) \to \mathrm{Rep}(\mathfrak{g}, \mathbb{C}),
\end{aligned}$$

*defined as follows. Given a complex representation $(V, \rho)$, we may regard $V$ as a real vector space and use the same $\rho$. Explicitly, this means to choose a $\mathbb{C}$-basis $\{e_1, \dots, e_n\}$ for $V$ and let $U_1 := \mathrm{span}_{\mathbb{R}}(e_1, \dots, e_n)$ and $U_2 := \mathrm{span}_{\mathbb{R}}(ie_1, \dots, ie_n)$. As a real vector space, $V$ is isomorphic to $U_1 \oplus U_2$. Let $\rho_{1ij}, \rho_{2ij} \in \mathbb{R}$ such that $\rho(x)e_i = \rho_{1ij}e_j + \rho_{2ij}ie_j$. As $\rho(x)$ is $\mathbb{C}$-linear, we have $\rho(x)ie_i = i\rho(x)e_i = -\rho_{2ij}e_j + \rho_{1ij}ie_j$. Written in the $\mathbb{R}$-basis, $\rho(x)$ acts on $U_1 \oplus U_2$ as $\tilde{\rho}(x) := \begin{bmatrix} \rho_1 & -\rho_2 \\ \rho_2 & \rho_1 \end{bmatrix}$. The functor $r_{\mathbb{R}}^{\mathbb{C}}$ takes $(V, \rho)$ to $(U_1 \oplus U_2, \tilde{\rho}(x))$. Similarly, we get $r_{\mathbb{C}}^{\mathbb{H}}$ and $r_{\mathbb{R}}^{\mathbb{H}}$. Additionally, given a representation of real or quaternionic type, we can forget about the structure map, giving us a complex representation. These are the functors $r_{\pm}$. Explicitly, the map $r_{\mathbb{C}}^{\mathbb{H}} = r_{-} \circ e_{-}$ and $r_{\mathbb{R}}^{\mathbb{H}} = r_{\mathbb{R}}^{\mathbb{C}} \circ r_{\mathbb{C}}^{\mathbb{H}}$.*

*We also have extension maps, obtained by extending the scalars on the vector space*

$$\begin{aligned}
&e_{\mathbb{R}}^{\mathbb{C}} \colon \mathrm{Rep}(\mathfrak{g}, \mathbb{R}) \to \mathrm{Rep}(\mathfrak{g}, \mathbb{C}),\\
&e_{\mathbb{C}}^{\mathbb{H}} \colon \mathrm{Rep}(\mathfrak{g}, \mathbb{C}) \to \mathrm{Rep}(\mathfrak{g}, \mathbb{H}),\\
&e_{\mathbb{R}}^{\mathbb{H}} \colon \mathrm{Rep}(\mathfrak{g}, \mathbb{R}) \to \mathrm{Rep}(\mathfrak{g}, \mathbb{H}), \quad e_{\mathbb{R}}^{\mathbb{H}} = e_{\mathbb{C}}^{\mathbb{H}} \circ e_{\mathbb{R}}^{\mathbb{C}},
\end{aligned}$$

*defined as follows. Given a real representation $(U, \rho)$, we take $r_{+} \circ e_{+}(U, \rho)$. The resulting representation is $e_{\mathbb{R}}^{\mathbb{C}}(U, \rho)$. We view $\mathbb{H}$ as a right $\mathbb{C}$-module via right multiplication $\mathbb{H} \times \mathbb{C} \to \mathbb{H}$ taking $(w, v) \mapsto wv$. Then, given a complex representation $(V, \rho)$, we have $e_{\mathbb{C}}^{\mathbb{H}}(V, \rho) := (\mathbb{H} \otimes_{\mathbb{C}} V, \mathrm{id}_{\mathbb{H}} \otimes \rho)$. Given a real representation $(U, \rho)$, we can do the same as above, taking $e_{\mathbb{R}}^{\mathbb{H}}(U, \rho) := (\mathbb{H} \otimes_{\mathbb{R}} U, \mathrm{id}_{\mathbb{H}} \otimes \rho)$. The result is the same as the composition of the two previous maps.*

*Finally, we have a conjugation map,*

$$c\colon \mathrm{Rep}(\mathfrak{g},\mathbb{C}) \to \mathrm{Rep}(\mathfrak{g},\mathbb{C}),$$

*defined as follows. Let $(V,\rho)$ be a complex representation. Let $\overline{V}$ be the vector space with the same additive structure as $V$ but scalar multiplication given by $\alpha \cdot v := \overline{\alpha} v$. Then $\rho(x)$ acts on $\overline{V}$ as $\overline{\rho(x)}$. Let $\overline{\rho}(x) := \overline{\rho(x)}$ and $c(V,\rho) := (\overline{V},\overline{\rho})$.*

**Proposition 1.10.** *The above functors satisfy the following relations*

$$\begin{aligned} r_{\mathbb{R}}^{\mathbb{C}} \circ e_{\mathbb{R}}^{\mathbb{C}}(U,\rho) &\simeq (U,\rho)^{\oplus 2}, & e_{\mathbb{R}}^{\mathbb{C}} \circ r_{\mathbb{R}}^{\mathbb{C}}(V,\rho) &\simeq (V,\rho) \oplus c(V,\rho),\\ r_{\mathbb{C}}^{\mathbb{H}} \circ e_{\mathbb{C}}^{\mathbb{H}}(V,\rho) &\simeq (V,\rho) \oplus c(V,\rho), & e_{\mathbb{C}}^{\mathbb{H}} \circ r_{\mathbb{C}}^{\mathbb{H}}(W,\rho) &\simeq (W,\rho)^{\oplus 2},\\ c \circ e_{\mathbb{R}}^{\mathbb{C}}(U,\rho) &\simeq e_{\mathbb{R}}^{\mathbb{C}}(U,\rho), & r_{\mathbb{R}}^{\mathbb{C}} \circ c(V,\rho) &\simeq r_{\mathbb{R}}^{\mathbb{C}}(V,\rho),\\ c \circ r_{\mathbb{C}}^{\mathbb{H}}(W,\rho) &\simeq r_{\mathbb{C}}^{\mathbb{H}}(W,\rho), & e_{\mathbb{C}}^{\mathbb{H}} \circ c(V,\rho) &\simeq e_{\mathbb{C}}^{\mathbb{H}}(V,\rho),\\ r_+ \circ e_+(U,\rho) &\simeq e_{\mathbb{R}}^{\mathbb{C}}(U,\rho), & r_- \circ e_-(W,\rho) &\simeq r_{\mathbb{C}}^{\mathbb{H}}(W,\rho), \end{aligned}$$
$$c^2(V,\rho) \simeq (V,\rho).$$

*Proof.* The first equivalency is obtained via the $\mathbb{R}$-linear isomorphism $\phi\colon \mathbb{R} \otimes_{\mathbb{R}} U \oplus i\mathbb{R} \otimes_{\mathbb{R}} U \to U \oplus U$ taking $\phi(a \otimes u, ib \otimes v) := (au, bv)$. We see that as $\rho(x)$ maps from a real vector space to itself,

$$\begin{aligned} \phi\left(\begin{bmatrix} 1 \otimes \rho(x) & 0 \\ 0 & 1 \otimes \rho(x) \end{bmatrix} \begin{bmatrix} a \otimes u \\ ib \otimes v \end{bmatrix}\right) &= (a\rho(x)u, ib\rho(x)v) \\ &= (\rho \oplus \rho)(x)\phi(a \otimes u, ib \otimes v). \end{aligned}$$

The second equivalency is obtained via the $\mathbb{C}$-linear isomorphism $\phi\colon V \oplus \overline{V} \to \mathbb{C} \otimes_{\mathbb{R}} (U_1 \oplus U_2)$ sending the basis elements of $V \oplus \overline{V}$ to

$$\begin{aligned} \phi(e_i, 0) &:= \frac{1 \otimes (e_i, 0) - i \otimes (0, ie_i)}{2}, \\ \phi(0, e_i) &:= \frac{1 \otimes (e_i, 0) + i \otimes (0, ie_i)}{2}. \end{aligned}$$

We see that

$$1 \otimes \begin{bmatrix} \rho_1 & -\rho_2 \\ \rho_2 & \rho_1 \end{bmatrix} \phi(e_i, 0) = \frac{1 \otimes (\rho_{1ij} e_j, \rho_{2ij} ie_j) - i \otimes (-\rho_{2ij} e_j, \rho_{1ij} ie_j)}{2} = \phi\left((\rho \oplus \overline{\rho})(x)(e_i, 0)\right),$$
$$1 \otimes \begin{bmatrix} \rho_1 & -\rho_2 \\ \rho_2 & \rho_1 \end{bmatrix} \phi(0, e_i) = \frac{1 \otimes (\rho_{1ij} e_j, \rho_{2ij} ie_j) + i \otimes (-\rho_{2ij} e_j, \rho_{1ij} ie_j)}{2} = \phi\left((\rho \oplus \overline{\rho})(x)(0, e_i)\right).$$

Note that in the final equality, we have that $\phi$ is $\mathbb{C}$-linear with respect to multiplication on $V \oplus \overline{V}$, so $\phi(0, \rho(x)_{ij} e_j) = \overline{\rho(x)_{ij}} \phi(0, e_i)$.

The third equivalency is obtained via the $\mathbb{C}$-linear isomorphism $\phi\colon (\mathbb{C} \otimes_{\mathbb{C}} V) \oplus (j\mathbb{C} \otimes_{\mathbb{C}} V) \to V \oplus \overline{V}$ taking

$$\begin{aligned} \phi(1 \otimes e_i, 0) &:= (e_i, 0), \\ \phi(0, j \otimes e_i) &:= (0, e_i). \end{aligned}$$

We see

$$\begin{aligned} \begin{bmatrix} \rho(x) & 0 \\ 0 & \overline{\rho(x)} \end{bmatrix} \cdot \phi(1 \otimes e_i, 0) &= (\rho(x) e_i, 0) = \rho(x)_{ij} \phi(1 \otimes e_j, 0) \\ &= \phi\left( \begin{bmatrix} 1 \otimes \rho(x) & 0 \\ 0 & 1 \otimes \overline{\rho(x)} \end{bmatrix} \begin{bmatrix} 1 \otimes e_i \\ 0 \end{bmatrix} \right), \\ \begin{bmatrix} \rho(x) & 0 \\ 0 & \overline{\rho(x)} \end{bmatrix} \cdot \phi(0, j \otimes e_i) &= (0, \rho(x) e_i) = \overline{\rho(x)_{ij}} \cdot (0, e_j) \\ &= \phi\left( \begin{bmatrix} 1 \otimes \rho(x) & 0 \\ 0 & 1 \otimes \overline{\rho(x)} \end{bmatrix} \begin{bmatrix} 0 & j \otimes e_i \end{bmatrix} \right). \end{aligned}$$

The fourth equivalency is obtained via the $\mathbb{H}$-linear isomorphism $\phi\colon W \oplus W \to \mathbb{H} \otimes_{\mathbb{C}} (V_1 \oplus V_2)$ sending the basis elements of $W \oplus W$ to

$$\begin{aligned} \phi(e_i, 0) &:= \frac{1 \otimes (e_i, 0) - j \otimes (0, j e_i)}{2}, \\ \phi(0, e_i) &:= \frac{i \otimes (e_i, 0) + k \otimes (0, j e_i)}{2}. \end{aligned}$$

We see that

$$\begin{aligned} 1 \otimes \begin{bmatrix} \rho_1 & -\overline{\rho_2} \\ \rho_2 & \overline{\rho_1} \end{bmatrix} \phi(e_i, 0) &= \frac{1 \otimes (\rho_{1ij} e_j, \rho_{2ij} j e_j) - j \otimes (-\overline{\rho_{2ij}} e_j, \overline{\rho_{1ij}} j e_j)}{2} \\ &= \rho(x)_{ij} \frac{1 \otimes (e_j, 0) - j \otimes (0, j e_j)}{2} = \phi((\rho \oplus \rho)(x)(e_i, 0)), \\ 1 \otimes \begin{bmatrix} \rho_1 & -\overline{\rho_2} \\ \rho_2 & \overline{\rho_1} \end{bmatrix} \phi(0, e_i) &= \frac{i \otimes (\rho_{1ij} e_j, \rho_{2ij} j e_j) + k \otimes (-\overline{\rho_{2ij}} e_j, \overline{\rho_{1ij}} j e_j)}{2} \\ &= \rho(x)_{ij} \frac{i \otimes (e_j, 0) + k \otimes (0, j e_j)}{2} = \phi((\rho \oplus \rho)(x)(0, e_i)). \end{aligned}$$

Note that we use that $\mathbb{H} \otimes_{\mathbb{C}} (V_1 \oplus V_2)$ is a $\mathbb{H}$-vector space, where we multiply on the $\mathbb{H}$ factor on the left. That is, $p(q \otimes (v, w)) = (pq) \otimes (v, w)$. Additionally, $\mathbb{H}$ is a right

$\mathbb{C}$-module, so if we have $\alpha \in \mathbb{C}$, we have $(q\alpha) \otimes (v,w) = q \otimes \alpha(v,w)$. Thus, we have $\alpha(j \otimes (v,w)) = (\alpha j) \otimes (v,w) = j \otimes \overline{\alpha}(v,w)$.

The fifth equivalency is obtained via the $\mathbb{C}$-linear isomorphism $\phi\colon \mathbb{C} \otimes_{\mathbb{R}} U \to \overline{\mathbb{C} \otimes_{\mathbb{R}} U}$ taking $\phi(\alpha \otimes u) := \overline{\alpha} \otimes u$. Indeed, we see

$$\phi(i\alpha \otimes u) = -i\phi(\alpha \otimes u) = i \cdot \phi(\alpha \otimes u).$$

Furthermore, we see that

$$\overline{1 \otimes \rho(x)} \cdot \phi(\alpha \otimes u) = \overline{\alpha} \otimes \rho(x)u = \phi((1 \otimes \rho(x))(\alpha \otimes u)).$$

The sixth equivalency is obtained via the $\mathbb{R}$-linear isomorphism $\phi\colon U \oplus iU \to U \oplus iU$ taking $\phi(u, iv) := (u, -iv)$. Let $\rho(x) = \rho_1 + i\rho_2$. Then $\overline{\rho(x)} = \rho_1 - i\rho_2$. Thus, the representation on the first $U \oplus iU$ is $\begin{bmatrix} \rho_1 & -\rho_2 \\ \rho_2 & \rho_1 \end{bmatrix}$ as it is constructed from $\rho$ and the representation on the second one is $\begin{bmatrix} \rho_1 & \rho_2 \\ -\rho_2 & \rho_1 \end{bmatrix}$, as it is constructed from $\overline{\rho}$. Then we see

$$\begin{bmatrix} \rho_1 & \rho_2 \\ -\rho_2 & \rho_1 \end{bmatrix} \phi(u, iv) = (\rho_1 u - \rho_2 v, -i\rho_2 u - i\rho_1 v) = \phi\left( \begin{bmatrix} \rho_1 & -\rho_2 \\ \rho_2 & \rho_1 \end{bmatrix} \begin{bmatrix} u \\ iv \end{bmatrix} \right).$$

The seventh equivalency is obtained via the $\mathbb{C}$-linear map $\phi\colon (\mathbb{C} \otimes_{\mathbb{C}} V) \oplus (j\mathbb{C} \otimes_{\mathbb{C}} V) \to \overline{\mathbb{C} \otimes_{\mathbb{C}} V} \oplus \overline{j\mathbb{C} \otimes_{\mathbb{C}} V}$ taking the basis vectors to

$$\phi(1 \otimes e_i, 0) := (0, -j \otimes e_i), \quad \text{and} \quad \phi(0, j \otimes e_i) := (1 \otimes e_i, 0).$$

We see

$$\begin{aligned}
\begin{bmatrix} 1 \otimes \overline{\rho_1(x)} & -1 \otimes \rho_2(x) \\ 1 \otimes \overline{\rho_2(x)} & 1 \otimes \rho_1(x) \end{bmatrix} \cdot \phi(1 \otimes e_i, 0) &= \rho_{1ij} \cdot (0, -j \otimes e_j) + \rho_{2ij} \cdot (1 \otimes e_j, 0) \\
&= \phi\left( \begin{bmatrix} 1 \otimes \rho_1(x) & -1 \otimes \overline{\rho_2(x)} \\ 1 \otimes \rho_2(x) & 1 \otimes \overline{\rho_1(x)} \end{bmatrix} \begin{bmatrix} 1 \otimes e_i \\ 0 \end{bmatrix} \right), \\
\begin{bmatrix} 1 \otimes \overline{\rho_1(x)} & -1 \otimes \rho_2(x) \\ 1 \otimes \overline{\rho_2(x)} & 1 \otimes \rho_1(x) \end{bmatrix} \cdot \phi(0, j \otimes e_i) &= \overline{\rho_{1ij}} \cdot (1 \otimes e_j, 0) + \overline{\rho_{2ij}} \cdot (0, j \otimes e_j) \\
&= \phi\left( \begin{bmatrix} 1 \otimes \rho_1(x) & -1 \otimes \overline{\rho_2(x)} \\ 1 \otimes \rho_2(x) & 1 \otimes \overline{\rho_1(x)} \end{bmatrix} \begin{bmatrix} 0 \\ j \otimes e_i \end{bmatrix} \right).
\end{aligned}$$

| $L =$ | $\mathbb{R}$ | $\mathbb{C}$ | $\mathbb{H}$ |
|---|---|---|---|
| $(U,\rho) \in \mathrm{Irr}(\mathfrak{g},\mathbb{R})_L$ | $e^{\mathbb{C}}_{\mathbb{R}}(U,\rho) \in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{R}}$ | $(U,\rho) = r^{\mathbb{C}}_{\mathbb{R}}(V,\lambda)$, $(V,\rho) \in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{C}}$ | $(U,\rho) = r^{\mathbb{C}}_{\mathbb{R}}(V,\lambda)$, $(V,\rho) \in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{H}}$ |
| $(V,\rho) \in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_L$ | $(V,\rho) \in \mathrm{Rep}_+(\mathfrak{g},\mathbb{C})$ | $(V,\rho) \not\simeq (\overline{V},\overline{\rho})$ | $(V,\rho) \in \mathrm{Rep}_-(\mathfrak{g},\mathbb{C})$ |
| $(W,\rho) \in \mathrm{Irr}(\mathfrak{g},\mathbb{H})_L$ | $(W,\rho) = e^{\mathbb{H}}_{\mathbb{C}}(V,\lambda)$, $(V,\lambda) \in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{R}}$ | $(W,\rho) = e^{\mathbb{H}}_{\mathbb{C}}(V,\lambda)$, $(V,\lambda) \in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{C}}$ | $r^{\mathbb{H}}_{\mathbb{C}}(W,\rho) \in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{H}}$ |

Table 1: This table outlines the various types of irreducible representations of $\mathfrak{g}$. In words, $(U,\rho) \in \mathrm{Irr}(\mathfrak{g},\mathbb{R})_{\mathbb{R}}$ if and only if $e^{\mathbb{C}}_{\mathbb{R}}(U,\rho) \in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{R}}$, which happens if and only if this representation is irreducible and of real type. The other definitions are similar.

The eighth equivalency is obtained via the $\mathbb{H}$-linear isomorphism $\phi\colon \mathbb{H} \otimes_{\mathbb{C}} V \to \mathbb{H} \otimes_{\mathbb{C}} \overline{V}$ taking basis vectors $\phi(1 \otimes e_i) := j \otimes e_i$. We see

$$(1 \otimes \overline{\rho(x)}) \cdot \phi(1 \otimes e_i) = j \otimes \overline{\rho(x)_{ij}} e_j = (\rho(x)_{ij} j) \otimes e_j = \phi((1 \otimes \rho(x))(1 \otimes e_i)).$$

The ninth and tenth equivalencies are just by definition.

The eleventh equivalency is obtained via the $\mathbb{C}$-linear isomorphism $\phi\colon V \to \overline{\overline{V}}$ taking $\phi(v) := v$. We see that

$$\rho(x) \cdot \phi(v) = \rho(x)v = \phi(\rho(x)v). \qquad \square$$

# 2 Irreducible representations of various types

In this section, we uncover the relationship between irreducible real, quaternionic, and complex representations. The main result is a classification of irreducible representations.

**Definition 2.1.** *Let* $\mathrm{Irr}(\mathfrak{g},K)$ *for* $K = \mathbb{R},\mathbb{C},\mathbb{H}$ *be the sets of irreducible representations over* $K$*. For each* $L = \mathbb{R},\mathbb{C},\mathbb{H}$*, we define a subset* $\mathrm{Irr}(\mathfrak{g},K)_L \subseteq \mathrm{Irr}(\mathfrak{g},K)$*. The definitions of these nine subsets are given in Table 1.*

*We call an irreducible representation (whether real, complex, or quaternionic) in a set with subscript* $\mathbb{R},\mathbb{C},\mathbb{H}$ *of* ***real, complex, or quaternionic type****, respectively. Thus, we extend our use of type from just complex representations.*

We show the following

**Theorem 2.2.** *For $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$, we have that $\mathrm{Irr}(\mathfrak{g}, K)$ is the disjoint union of its subsets $\mathrm{Irr}(\mathfrak{g}, K)_{\mathbb{R}}, \mathrm{Irr}(\mathfrak{g}, K)_{\mathbb{C}}, \mathrm{Irr}(\mathfrak{g}, K)_{\mathbb{H}}$. Note that some of these subsets can be empty.*

First, we need some lemmas to simplify our proof.

**Lemma 2.3.** *A complex representation $(V, \rho)$ is of real (resp. quaternionic) type if and only if there exists a nonsingular symmetric (resp. skew-symmetric) $\mathfrak{g}$-invariant bilinear form $B\colon V \times V \to \mathbb{C}$.*

*Proof.* Suppose that such a $B\colon V \times V \to \mathbb{C}$ is given. Then $B(v, w) = \epsilon B(w, v)$ for $\epsilon = \pm 1$. Choose a $\mathfrak{g}$-invariant Hermitian inner product $\langle \cdot, \cdot \rangle$ on $V$ and define $f\colon V \to V$ by requiring $B(v, w) = \langle v, f(w) \rangle$ for all $v \in V$. Hence, $f$ is conjugate-linear. Indeed, for all $v \in V$,

$$\langle v, \overline{\alpha} f(w) - f(\alpha w) \rangle = \alpha B(v, w) - B(v, \alpha w) = 0.$$

Hence, $f(\alpha w) = \overline{\alpha} f(w)$. Furthermore, we see that as $B(\rho(x)v, w) = -B(v, \rho(x)w)$ and $\langle \rho(x)v, w \rangle = -\langle v, \rho(x)w \rangle$,

$$\begin{aligned} \langle v, \rho(x) f(w) \rangle &= -\langle \rho(x)v, f(w) \rangle = -B(\rho(x)v, w) \\ &= B(v, \rho(x)w) = \langle v, f(\rho(x)w) \rangle. \end{aligned}$$

Thus, $f(\rho(x)w) = \rho(x) f(w)$, so $f$ is equivariant. Finally, suppose $f(w) = 0$. Then for all $v \in V$,

$$B(v, w) = \langle v, f(w) \rangle = 0.$$

Thus, $w = 0$, so $f$ is an isomorphism (between $(V, \rho)$ and $(\overline{V}, \overline{\rho})$).

We see that

$$\langle v, f(w) \rangle = B(v, w) = \epsilon B(w, v) = \epsilon \langle w, f(v) \rangle = \epsilon \overline{\langle f(v), w \rangle}.$$

Hence,

$$\langle v, f^2(w) \rangle = \epsilon \overline{\langle f(v), f(w) \rangle} = \epsilon^2 \langle f^2(v), w \rangle = \langle f^2(v), w \rangle.$$

Therefore, we see that $\epsilon f^2$ is Hermitian and positive-definite. Indeed,

$$\langle v, \epsilon f^2(v) \rangle = \epsilon^2 \overline{\langle f(v), f(v) \rangle} \geq 0. \tag{1}$$

If $\langle v, \epsilon f^2(v) \rangle = 0$, then we see that $f(v) = 0$, so $v = 0$. Therefore, $V$ can be decomposed into the direct sum of eigenspaces $V_\lambda$ of $\epsilon f^2$. As $\epsilon f^2$ is positive-definite and Hermitian, we have all the eigenvalues are positive real numbers.

Let $v \in V_\lambda$. Then

$$\begin{aligned} \epsilon f^2(\rho(x)v) &= \rho(x) \epsilon f^2(v) = \lambda \rho(x) v, \\ \epsilon f^2(f(v)) &= f(\epsilon f^2(v)) = f(\lambda v) = \lambda f(v). \end{aligned}$$

Thus, the eigenspaces are $\mathfrak{g}$-invariant and invariant under $f$. Define $h\colon V \to V$ by taking $v \in V_\lambda$ to $h(v) := \sqrt{\lambda}v$ and extend linearly (take a basis of each eigenspace to get a basis on $V$). We see that for all $v \in V_\lambda$, as $f(v) \in V_\lambda$ as well,

$$h(f(v)) = \sqrt{\lambda}f(v) = f(\sqrt{\lambda}v) = f(h(v).$$

Furthermore, we see that for all $v \in V_\lambda$, as $\rho(x)v \in V_\lambda$ as well,

$$h(\rho(x)v) = \sqrt{\lambda}\rho(x)v = \rho(x)h(v).$$

Hence, $h$ is equivariant. Additionally, for all $v \in V_\lambda$,

$$h^2(v) = \lambda v = \epsilon f^2(v).$$

Therefore, $h^2 = \epsilon f^2$.

Define $J\colon V \to V$ by $J := h \circ f^{-1}$. As $f$ is conjugate-linear and $h$ linear, we have $J$ is conjugate-linear. Furthermore, we see from above that

$$J(\rho(x)v) = h(f^{-1}(\rho(x)v)) = h(\rho(x)f^{-1}(v)) = \rho(x)J(v).$$

Thus, $J$ is equivariant. Finally, we see that

$$J^2 = h \circ f^{-1} \circ h \circ f^{-1} = h^2 \circ f^{-2} = \epsilon \cdot \mathrm{id}_V.$$

Therefore, if $\epsilon = 1$, then we have a real structure and if $\epsilon = -1$, then we have a quaternionic structure.

Conversely, suppose $V$ has a structure map $J\colon V \to V$ such that $J^2 = \epsilon \cdot \mathrm{id}_V$. If $\epsilon = 1$, then $V \simeq \mathbb{C} \otimes_{\mathbb{R}} V_+$, where $V_+$ is the $+1$-eigenspace of $J$. Any non-singular, $\mathfrak{g}$-invariant, symmetric, $\mathbb{R}$-bilinear form on $V_+$ can be extended to a $\mathbb{C}$-bilinear form $B$ on $V$, which is still non-singular, $\mathfrak{g}$-invariant, and symmetric.

If $\epsilon = -1$, then consider $V$ as a $\mathbb{H}$-module, where the action of $j$ being that of $J$. Then $V$ carries a $\mathfrak{g}$-invariant, symplectic inner product $\langle \cdot, \cdot \rangle$. Write

$$\langle u, v \rangle = H(u, v) + B(u, v)j,$$

where $H(u, v), B(u, v) \in \mathbb{C}$. As $\lambda\langle u, v \rangle = \langle \lambda u, v \rangle$ and $\langle u, \lambda v \rangle = \langle u, v \rangle\overline{\lambda}$ for $\lambda \in \mathbb{H}$, we see that for $\alpha \in \mathbb{C}$,

$$\begin{aligned} H(u, \alpha v + w) + B(u, \alpha v + w)j &= \langle u, \alpha v + w \rangle = \langle u, w \rangle + \langle u, v \rangle\overline{\alpha} \\ &= H(u, w) + H(u, v)\overline{\alpha} + B(u, w)j + B(u, v)j\overline{\alpha}. \end{aligned}$$

Noting that $j\overline{\alpha} = \alpha j$ and

$$\begin{aligned} H(\alpha u + v, w) + B(\alpha u + v, w)j &= \langle \alpha u + v, w \rangle = \alpha\langle u, w \rangle + \langle v, w \rangle \\ &= \alpha H(u, w) + H(v, w) + \alpha B(u, w)j + B(v, w)j, \end{aligned}$$

we see that $B$ is $\mathbb{C}$-bilinear.

As $\langle u, v\rangle = \overline{\langle v, u\rangle}$, we have

$$\begin{aligned} H(u,v) + B(u,v)j = \langle u, v\rangle = \overline{\langle v, u\rangle} &= \overline{H(v,u)} + \overline{B(v,u)j} \\ &= \overline{H(v,u)} - B(v,u)j. \end{aligned}$$

Hence, $B$ is skew-symmetric.

Suppose $B(u, v) = 0$ for all $u \in V$. Then $\langle u, v\rangle \in \mathbb{C}$ for all $u \in V$. As,

$$\langle ju, v\rangle = j\langle u, v\rangle,$$

we see that $\langle u, v\rangle = 0$ for all $u \in V$. As symplectic inner products are non-singular, we have $v = 0$, so $B$ is non-singular. Finally, we see that

$$\begin{aligned} H(\rho(x)u, v) + B(\rho(x)u, v)j &= \langle \rho(x)u, v\rangle = -\langle u, \rho(x)v\rangle \\ &= -H(u, \rho(x)v) - B(u, \rho(x)v)j. \end{aligned}$$

Thus, $B$ is $\mathfrak{g}$-invariant. □

**Definition 2.4.** *We say a complex representation is **self-dual** if* $(V, \rho) \simeq (\overline{V}, \overline{\rho})$.

**Definition 2.5.** *Suppose* $\mathfrak{g}$ *is a semi-simple Lie algebra with representation* $(V, \rho)$*. The following representation* $(V^*, \rho^*)$ *is known as the **dual representation**. Here* $V^* := \operatorname{Hom}(V, \mathbb{C})$ *and* $\rho^*(x)f(v) := -f(\rho(x)v)$.

**Lemma 2.6.** *The pair* $(V^*, \rho^*)$ *is indeed a representation.*

*Proof.* We see that

$$\begin{aligned} ([\rho^*(x), \rho^*(y)]f)(v) &= -\rho^*(y)f(\rho(x)v) + \rho^*(x)f(\rho(y)v) = f([\rho(y), \rho(x)]v) \\ &= -f(\rho([x,y])v) = \rho^*([x,y])f(v) \end{aligned}$$

Therefore, we have a representation. □

Recall the definition of the representation $(\overline{V}, \overline{\rho})$ given in Definition 1.9.

**Lemma 2.7.** *Given an* $\mathfrak{g}$*-invariant inner product on* $V$*, we have that* $(V^*, \rho^*) \simeq (\overline{V}, \overline{\rho})$.

*Proof.* Consider $\phi \colon \overline{V} \to V^*$ taking $\phi(v) \mapsto \langle \cdot, v\rangle$. This map is a $\mathbb{C}$-linear isomorphism. Furthermore, we see that

$$\begin{aligned} \phi(\overline{\rho}(x) \cdot v)(w) = \phi(\rho(x)v)(w) &= \langle w, \rho(x)v\rangle = -\langle \rho(x)w, v\rangle \\ &= -\phi(v)(\rho(x)w) = \rho^*(x)\phi(v)(w). \end{aligned}$$

□

**Lemma 2.8.** *Given a representation* $(V, \rho)$, $((V^*)^*, (\rho^*)^*)$ *is isomorphic to* $(V, \rho)$.

*Proof.* Let $\{v_1, \ldots, v_n\}$ be a basis for $V$ and $\langle \cdot, \cdot \rangle$ an inner product on $V$. Construct $f\colon V \to (V^*)^*$ taking $v \mapsto (v^*)^*$. We see that for all $x \in \mathfrak{g}$ and $\phi \in V^*$ we have

$$\begin{aligned}(\rho^*)^*(x)f(v)(\phi) &= (\rho^*)^*(x)(v^*)^*(\phi) = -(v^*)^*(\rho^*(x)\phi) \\ &= -(\rho^*(x)\phi)(v) = \phi(\rho(x)v) = f(\rho(x)v)(\phi).\end{aligned}$$

Thus, the representations are isomorphic. □

**Lemma 2.9.** *Let* $(V, \rho) \in \operatorname{Irr}(\mathfrak{g}, \mathbb{C})$ *be self-dual. Then it is of real or quaternionic type, but not both.*

*Proof.* Let $B\colon V \to V^* \simeq \overline{V}$ give an equivalence $(V, \rho) \simeq (\overline{V}, \overline{\rho}) \simeq (V^*, \rho^*)$. That is, $B$ is equivariant and a linear isomorphism. We can think of $B$ as a non-singular, bilinear form on $V$. Consider $B^{\pm} := B \pm B^T$, where $B^T(u, v) := B(v, u)$. At least one of these is non-zero, or else $B = 0$.

Notice that

$$\begin{aligned}B^+(v, u) &= B(v, u) + B(u, v) = B^+(u, v), \\ B^-(v, u) &= B(v, u) - B(u, v) = -B^-(u, v).\end{aligned}$$

Thus, $B^+$ is symmetric and $B^-$ is skew-symmetric. Furthermore, as $B$ is equivariant, we have that $B(\rho(x)u) = \rho^*(x)B(u)$. Evaluating at $v$, we see

$$B(\rho(x)u, v) = -B(u, \rho(x)v).$$

Thus, $B$ is $\mathfrak{g}$-invariant.

Suppose $B^+$ is non-zero. Then we have a nonsingular, symmetric, bilinear, $\mathfrak{g}$-invariant inner product on $V$. By Lemma 2.3, $(V, \rho)$ is of real type.

Suppose $B^-$ is non-zero. Then we have a nonsingular, skew-symmetric, bilinear, $\mathfrak{g}$-invariant inner product on $V$. By Lemma 2.3, $(V, \rho)$ is of quaternionic type.

Suppose $(V, \rho)$ is of both types. Then there exists nonsingular, bilinear, $\mathfrak{g}$-invariant inner products $A, C$ on $V$, that are symmetric and skew-symmetric, respectively. Let $f\colon V \to V^*$ and $g\colon V \to V^*$ be given by $f(u)(v) := A(u, v)$ and $g(u)(v) := C(u, v)$. As $A$ and $C$ are nonsingular and bilinear, $f$ and $g$ are isomorphisms. Furthermore, we see

$$f(\rho(x)u)(v) = A(\rho(x)u, v) = -A(u, \rho(x)v) = \rho^*(x)f(u)(v).$$

Thus, $f$ is equivariant, and $g$ is similarly. By Schur's Lemma, any two isomorphisms are non-zero multiples of each other, so $A = \alpha C$ for some $\alpha \neq 0$. Hence, $A$ and $C$ are both symmetric and skew-symmetric, so they are both zero, contradiction! □

We are now able to prove Theorem 2.2.

*Proof of Theorem 2.2.* We divide the proof into three parts, one for each $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Let $(V, \rho) \in \mathrm{Irr}(\mathfrak{g}, \mathbb{C})$. If the representation is of real or quaternionic type, then $J\colon V \to V$ is a structure map with $J^2 = \pm \mathrm{id}_V$. In either case, $J$ is conjugate-linear, so thinking of it as a map $V \to \overline{V}$, we see that it is linear. Furthermore, we have that $J$ is equivariant, so

$$J(\rho(x)v) = \rho(x)J(v) = \overline{\rho}(x) \cdot J(v).$$

As $J^2 = \pm \mathrm{id}$, we have that it is bijective. Therefore, as a map $V \to \overline{V}$, $J$ is an isomorphism and it provides an equivalence $(V, \rho) \simeq (\overline{V}, \overline{\rho})$. Hence, if the representation is of real or quaternionic type, it cannot be of complex type.

Suppose $(V, \rho)$ is self-dual. By Lemma 2.9, we see that $(V, \rho)$ is of real or quaternionic type, but not both. Otherwise, $(V, \rho)$ is not self-dual and is of complex type.

Let $(U, \rho) \in \mathrm{Irr}(\mathfrak{g}, \mathbb{R})$. Denote $e := e^{\mathbb{C}}_{\mathbb{R}}$ and $r := r^{\mathbb{C}}_{\mathbb{R}}$. As $e = r_+ \circ e_+$, we have $e(U, \rho)$ is of real type, as there is a real structure on $e(U, \rho)$. As such, if $e(U, \rho)$ is irreducible, then $e(U, \rho) \in \mathrm{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{R}}$, so $(U, \rho) \in \mathrm{Irr}(\mathfrak{g}, \mathbb{R})_{\mathbb{R}}$. If $e(U, \rho)$ is not irreducible, let us decompose it in terms of irreducible summands as $e(U, \rho) = \bigoplus_{j=1}^{t} (V_j, \rho_j)$, where $(V_j, \rho_j) \in \mathrm{Irr}(\mathfrak{g}, \mathbb{C})$ and $t \geq 2$. Then

$$(U, \rho) \oplus (U, \rho) = r \circ e(U, \rho) = \bigoplus_{j=1}^{t} r(V_j, \rho_j).$$

As $t \geq 2$ and $(U, \rho)$ is irreducible, we have $t = 2$ and $r(V_j, \rho_j) = (U, \rho)$ for $j = 1, 2$. Then for $j = 1, 2$

$$(V_1, \rho_1) \oplus (V_2, \rho_2) = e(U, \rho) = e \circ r(V_j, \rho_j) = (V_j, \rho_j) \oplus (\overline{V_j}, \overline{\rho_j}).$$

Thus, we must have $(\overline{V_1}, \overline{\rho_1}) \simeq (V_2, \rho_2)$ and vice versa.

If $(V_1, \rho_1)$ is not self-dual, then we see $(U, \rho) \in \mathrm{Irr}(\mathfrak{g}, \mathbb{R})_{\mathbb{C}}$. Otherwise, we have a self-dual, irreducible, complex representation. Hence, $(V_1, \rho_1)$ is of either real or quaternionic type. If it is of real type, then $(V_1, \rho_1) \in \mathrm{Rep}_+(\mathfrak{g}, \mathbb{C})$ so $(X, \lambda) := s_+(V_1, \rho_1) \in \mathrm{Rep}(\mathfrak{g}, \mathbb{R})$. But then $e_+(X, \lambda) = e_+ \circ s_+(V_1, \rho_1) \simeq (V_1, \rho_1)$, so

$$e(X, \lambda) = r_+ \circ e_+(X, \lambda) \simeq r_+(V_1, \rho_1) = (V_1, \rho_1).$$

Thus,

$$(U, \rho) = r(V_1, \rho_1) \simeq r \circ e(X, \lambda) = (X, \lambda) \oplus (X, \lambda).$$

But $(U, \rho)$ is irreducible, contradiction! Thus, $(V_1, \rho_1)$ is of quaternionic type and $(U, \rho) \in \mathrm{Irr}(\mathfrak{g}, \mathbb{R})_{\mathbb{H}}$.

Note that $(U, \rho)$ cannot be in the intersection of any two of these sets. If it is the restriction of an irreducible complex representation, that representation cannot be both not self-dual and of quaternionic type. Furthermore, if $(U, \rho)$ is the restriction of an irreducible complex representation $(V, \lambda)$, then $e(U, \rho) = (V, \lambda) \oplus (\overline{V}, \overline{\lambda})$, which is not irreducible.

Finally, let $(W, \rho) \in \mathrm{Irr}(\mathfrak{g}, \mathbb{H})$. Let $r := r_{\mathbb{C}}^{\mathbb{H}}$ and $e := e_{\mathbb{C}}^{\mathbb{H}}$. As $r = r_- \circ e_-$, $r(W, \rho)$ has a quaternionic structure. As such, if $r(W, \rho)$ is irreducible, then $r(W, \rho) \in \mathrm{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{H}}$, so $(W, \rho) \in \mathrm{Irr}(\mathfrak{g}, \mathbb{H})_{\mathbb{H}}$. If $r(W, \rho)$ is not irreducible, let us decompose it in terms of irreducible summands as $r(W, \rho) = \bigoplus_{j=1}^{t} (V_j, \rho_j)$, where $(V_j, \rho_j) \in \mathrm{Irr}(\mathfrak{g}, \mathbb{C})$ and $t \geq 2$. Then

$$(W, \rho) \oplus (W, \rho) = e \circ r(W, \rho) = \bigoplus_{j=1}^{t} e(V_j, \rho_j).$$

As $t \geq 2$ and $(W, \rho)$ is irreducible, we have $t = 2$ and $e(V_j, \rho_j) = (W, \rho)$ for $j = 1, 2$. Then for $j = 1, 2$

$$(V_1, \rho_1) \oplus (V_2, \rho_2) = r(W, \rho) = r \circ e(V_j, \rho_j) = (V_j, \rho_j) \oplus (\overline{V_j}, \overline{\rho_j}).$$

Thus, we must have $(\overline{V_1}, \overline{\rho_1}) \simeq (V_2, \rho_2)$ and vice versa.

If $(V_1, \rho_1)$ is not self-dual, then we see that $(W, \rho) \in \mathrm{Irr}(\mathfrak{g}, \mathbb{H})_{\mathbb{C}}$. Otherwise, we have a self-dual, irreducible, complex representation. Hence, $(V_1, \rho_1)$ is of either real or quaternionic type. If it is of quaternionic type, then $(V_1, \rho_1) \in \mathrm{Rep}_-(\mathfrak{g}, \mathbb{C})$, so $(X, \lambda) := s_-(V_1, \rho_1) \in \mathrm{Rep}(\mathfrak{g}, \mathbb{H})$. But then $e_-(X, \lambda) = e_- \circ s_-(V_1, \rho_1) \simeq (V_1, \rho_1)$, so

$$r(X, \lambda) = r_- \circ e_-(X, \lambda) \simeq r_-(V_1, \rho_1) = (V_1, \rho_1).$$

Thus,

$$(W, \rho) = e(V_1, \rho_1) \simeq e \circ r(X, \lambda) = (X, \lambda) \oplus (X, \lambda).$$

But $(W, \rho)$ is irreducible, contradiction! Therefore, $(V_1, \rho_1)$ is of real type and $(W, \rho) \in \mathrm{Irr}(\mathfrak{g}, \mathbb{H})_{\mathbb{R}}$.

Note that $(W, \rho)$ cannot be in the intersection of any two of these sets. If it is the extension of an irreducible complex representation, that representation cannot be both not self-dual and of real type. Furthermore, if $(W, \rho)$ is the extension of an irreducible complex representation $(V, \lambda)$, then $r(W, \rho) = (V, \lambda) \oplus (\overline{V}, \overline{\lambda})$, which is not irreducible. □

We have proven the first six of the following implications. The remaining three are proven similarly to each other.

**Proposition 2.10.** *We have the following relationships between irreducible representations.*

$$
\begin{array}{llcll}
(1) & (U,\rho)\in \mathrm{Irr}(\mathfrak{g},\mathbb{R})_{\mathbb{R}} & \Rightarrow & e_{\mathbb{R}}^{\mathbb{C}}(U,\rho)=(V,\lambda), & \text{for } (V,\lambda)\in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{R}},\\
(2) & (U,\rho)\in \mathrm{Irr}(\mathfrak{g},\mathbb{R})_{\mathbb{C}} & \Rightarrow & e_{\mathbb{R}}^{\mathbb{C}}(U,\rho)=(V,\lambda)\oplus(\overline{V},\overline{\lambda}), & \text{for } (V,\lambda)\in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{C}},\\
(3) & (U,\rho)\in \mathrm{Irr}(\mathfrak{g},\mathbb{R})_{\mathbb{H}} & \Rightarrow & e_{\mathbb{R}}^{\mathbb{C}}(U,\rho)=(V,\lambda)\oplus(V,\lambda), & \text{for } (V,\lambda)\in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{H}},\\
(4) & (W,\rho)\in \mathrm{Irr}(\mathfrak{g},\mathbb{H})_{\mathbb{R}} & \Rightarrow & r_{\mathbb{C}}^{\mathbb{H}}(W,\rho)=(V,\lambda)\oplus(V,\lambda), & \text{for } (V,\lambda)\in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{R}},\\
(5) & (W,\rho)\in \mathrm{Irr}(\mathfrak{g},\mathbb{H})_{\mathbb{C}} & \Rightarrow & r_{\mathbb{C}}^{\mathbb{H}}(W,\rho)=(V,\lambda)\oplus(\overline{V},\overline{\lambda}), & \text{for } (V,\lambda)\in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{C}},\\
(6) & (W,\rho)\in \mathrm{Irr}(\mathfrak{g},\mathbb{H})_{\mathbb{H}} & \Rightarrow & r_{\mathbb{C}}^{\mathbb{H}}(W,\rho)=(V,\lambda), & \text{for } (V,\lambda)\in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{H}},\\
(7) & (V,\rho)\in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{R}} & \Rightarrow & \begin{array}{c} r_{\mathbb{R}}^{\mathbb{C}}(V,\rho)=(U,\lambda)\oplus(U,\lambda), \\ e_{\mathbb{C}}^{\mathbb{H}}(V,\rho)=(W,\lambda), \end{array} & \begin{array}{l} \text{for } (U,\lambda)\in \mathrm{Irr}(\mathfrak{g},\mathbb{R})_{\mathbb{R}}, \\ \text{for } (W,\lambda)\in \mathrm{Irr}(\mathfrak{g},\mathbb{H})_{\mathbb{R}}, \end{array}\\
(8) & (V,\rho)\in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{C}} & \Rightarrow & \begin{array}{c} r_{\mathbb{R}}^{\mathbb{C}}(V,\rho)=(U,\lambda)=r_{\mathbb{R}}^{\mathbb{C}}(\overline{V},\overline{\rho}), \\ e_{\mathbb{C}}^{\mathbb{H}}(V,\rho)=(W,\lambda)=e_{\mathbb{C}}^{\mathbb{H}}(\overline{V},\overline{\rho}), \end{array} & \begin{array}{l} \text{for } (U,\lambda)\in \mathrm{Irr}(\mathfrak{g},\mathbb{R})_{\mathbb{C}}, \\ \text{for } (W,\lambda)\in \mathrm{Irr}(\mathfrak{g},\mathbb{H})_{\mathbb{C}}, \end{array}\\
(9) & (V,\rho)\in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{H}} & \Rightarrow & \begin{array}{c} r_{\mathbb{R}}^{\mathbb{C}}(V,\rho)=(U,\lambda), \\ e_{\mathbb{C}}^{\mathbb{H}}(V,\rho)=(W,\lambda)\oplus(W,\lambda), \end{array} & \begin{array}{l} \text{for } (U,\lambda)\in \mathrm{Irr}(\mathfrak{g},\mathbb{R})_{\mathbb{H}}, \\ \text{for } (W,\lambda)\in \mathrm{Irr}(\mathfrak{g},\mathbb{H})_{\mathbb{H}}. \end{array}
\end{array}
$$

*Proof.* (1) This is the definition of being in $\mathrm{Irr}(\mathfrak{g},\mathbb{R})_{\mathbb{R}}$.

(2) By definition, we have $(U,\rho)=r_{\mathbb{R}}^{\mathbb{C}}(V,\lambda)$ for some $(V,\lambda)\in\mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{C}}$. By Proposition 1.10, $e_{\mathbb{R}}^{\mathbb{C}}\circ r_{\mathbb{R}}^{\mathbb{C}}\simeq 1\oplus c$.

(3) By definition, we have $(U,\rho)=r_{\mathbb{R}}^{\mathbb{C}}(V,\lambda)$ for some $(V,\lambda)\in\mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{H}}$. By Proposition 1.10, $e_{\mathbb{R}}^{\mathbb{C}}\circ r_{\mathbb{R}}^{\mathbb{C}}\simeq 1\oplus c$ and we note that representations of quaternionic type are self-dual.

(4) By definition, we have $(W,\rho)=e_{\mathbb{C}}^{\mathbb{H}}(V,\lambda)$ for some $(V,\lambda)\in\mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{R}}$. By Proposition 1.10, $r_{\mathbb{C}}^{\mathbb{H}}\circ e_{\mathbb{C}}^{\mathbb{H}}\simeq 1\oplus c$ and we note that representations of real type are self-dual.

(5) By definition, we have $(W,\rho)=e_{\mathbb{C}}^{\mathbb{H}}(V,\lambda)$ for some $(V,\lambda)\in\mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{C}}$. By Proposition 1.10, $r_{\mathbb{C}}^{\mathbb{H}}\circ e_{\mathbb{C}}^{\mathbb{H}}\simeq 1\oplus c$.

(6) This is the definition of being in $\mathrm{Irr}(\mathfrak{g},\mathbb{H})_{\mathbb{H}}$.

(7) Suppose $(V,\rho)\in\mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{R}}$. Then it belongs to $\mathrm{Rep}_+(\mathfrak{g},\mathbb{C})$, so let $(U,\lambda):=s_+(V,\rho)\in\mathrm{Rep}(\mathfrak{g},\mathbb{R})$. We have that $e_+(U,\lambda)\simeq(V,\rho)$, so

$$e_{\mathbb{R}}^{\mathbb{C}}(U,\lambda)=r_+\circ e_+(U,\lambda)\simeq r_+(V,\rho)=(V,\rho).$$

Thus,

$$r_{\mathbb{R}}^{\mathbb{C}}(V,\rho)\simeq r_{\mathbb{R}}^{\mathbb{C}}\circ e_{\mathbb{R}}^{\mathbb{C}}(U,\lambda)=(U,\lambda)\oplus(U,\lambda).$$

Let us decompose $(U,\lambda)$ as $(U,\lambda)=\bigoplus_{j=1}^{t}(U_j,\lambda_j)$, where $t\geq 1$ and $(U_j,\lambda_j)\in$

$\mathrm{Irr}(\mathfrak{g}, \mathbb{R})$. We see that

$$(V, \rho) \simeq \bigoplus_{j=1}^{t} e_{\mathbb{R}}^{\mathbb{C}}(U_j, \lambda_j).$$

As $(V, \rho)$ is irreducible, we have $t = 1$, so $(U, \lambda) \in \mathrm{Irr}(\mathfrak{g}, \mathbb{R})_{\mathbb{R}}$, proving the first part.

Let $(W, \lambda) := e_{\mathbb{C}}^{\mathbb{H}}(V, \rho)$. If we prove that $(W, \lambda)$ is irreducible, then we have proven the result. Suppose not, then let us decompose it as $(W, \lambda) = \bigoplus_{j=1}^{t}(W_j, \lambda_j)$, where $t \geq 2$ and $(W_j, \lambda_j) \in \mathrm{Irr}(\mathfrak{g}, \mathbb{H})$. We see that as $(V, \rho)$ is of real type,

$$(V, \rho) \oplus (V, \rho) = r_{\mathbb{C}}^{\mathbb{H}} \circ e_{\mathbb{C}}^{\mathbb{H}}(V, \rho) = \bigoplus_{j=1}^{t} r_{\mathbb{C}}^{\mathbb{H}}(W_j, \lambda_j).$$

As $t \geq 2$ and $(V, \rho)$ is irreducible, we see that $t = 2$ and $r_{\mathbb{C}}^{\mathbb{H}}(W_j, \lambda_j) = (V, \rho)$ for $j = 1, 2$. We know from above that if $(W_j, \rho_j)$ is of real or complex type, then $(V, \rho) = r_{\mathbb{C}}^{\mathbb{H}}(W_j, \rho_j)$ is not irreducible, but it is. Thus, $(W_j, \rho_j)$ must be of quaternionic type. But then $(V, \rho) = r_{\mathbb{C}}^{\mathbb{H}}(W_j, \rho_j)$ is of real and quaternionic type, which is not possible. Contradiction! Therefore, $(W, \lambda)$ is irreducible.

(8) Suppose $(V, \rho) \in \mathrm{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{C}}$. Let $(U, \lambda) := r_{\mathbb{R}}^{\mathbb{C}}(V, \rho)$. If $(U, \rho)$ is irreducible, then we have proven the result, as

$$r_{\mathbb{R}}^{\mathbb{C}}(\overline{V}, \overline{\rho}) = r_{\mathbb{R}}^{\mathbb{C}} \circ c(V, \rho) = r_{\mathbb{R}}^{\mathbb{C}}(V, \rho) = (U, \lambda).$$

Suppose not, then let us decompose it as $(U, \lambda) = \bigoplus_{j=1}^{t}(U_j, \lambda_j)$, where $t \geq 2$ and $(U_j, \lambda_j) \in \mathrm{Irr}(\mathfrak{g}, \mathbb{R})$. We see that

$$(V, \rho) \oplus (\overline{V}, \overline{\rho}) = e_{\mathbb{R}}^{\mathbb{C}} \circ r_{\mathbb{R}}^{\mathbb{C}}(V, \rho) = \bigoplus_{j=1}^{t} e_{\mathbb{R}}^{\mathbb{C}}(U_j, \lambda_j).$$

As $(V, \rho)$ and its dual are irreducible (see the Dual Representation document) and $t \geq 2$, we have $t = 2$, so without loss of generality, $e_{\mathbb{R}}^{\mathbb{C}}(U_1, \lambda_1) = (V, \rho)$ and $e_{\mathbb{R}}^{\mathbb{C}}(U_2, \lambda_2) = (\overline{V}, \overline{\rho})$. As $(U_1, \lambda_1)$ is irreducible, if it is of complex or quaternionic type, then $(V, \rho) = e_{\mathbb{R}}^{\mathbb{C}}(U_1, \rho_1)$ is not irreducible, but it is. Thus, $(U_1, \rho_1)$ must be of real type. But then $(V, \rho) = e_{\mathbb{R}}^{\mathbb{C}}(U_1, \rho_1)$ is of real and complex type, which is not possible. Contradiction! Therefore, $(U, \lambda)$ is irreducible.

Let $(W, \lambda) := e_{\mathbb{C}}^{\mathbb{H}}(V, \rho)$. If $(W, \rho)$ is irreducible, then we have proven the result, as

$$e_{\mathbb{C}}^{\mathbb{H}}(\overline{V}, \overline{\rho}) = e_{\mathbb{C}}^{\mathbb{H}} \circ c(V, \rho) = e_{\mathbb{C}}^{\mathbb{H}}(V, \rho) = (W, \lambda).$$

Suppose not, then let us decompose it as $(W,\lambda) = \bigoplus_{j=1}^{t}(W_j,\lambda_j)$, where $t \geq 2$ and $(W_j,\lambda_j) \in \mathrm{Irr}(\mathfrak{g},\mathbb{H})$. We see that

$$(V,\rho) \oplus (\overline{V},\overline{\rho}) = r_{\mathbb{C}}^{\mathbb{H}} \circ e_{\mathbb{C}}^{\mathbb{H}}(V,\rho) = \bigoplus_{j=1}^{t} r_{\mathbb{C}}^{\mathbb{H}}(W_j,\lambda_j).$$

As $(V,\rho)$ and its dual are irreducible (see the Dual Representation document) and $t \geq 2$, we have $t = 2$, so without loss of generality, $r_{\mathbb{C}}^{\mathbb{H}}(W_1,\lambda_1) = (V,\rho)$ and $r_{\mathbb{C}}^{\mathbb{H}}(W_2,\lambda_2) = (\overline{V},\overline{\rho})$. As $(W_1,\lambda_1)$ is irreducible, if it is of complex or real type, then $(V,\rho) = r_{\mathbb{C}}^{\mathbb{H}}(W_1,\rho_1)$ is not irreducible, but it is. Thus, $(W_1,\rho_1)$ must be of quaternionic type. But then $(V,\rho) = r_{\mathbb{C}}^{\mathbb{H}}(U_1,\rho_1)$ is of quaternionic and complex type, which is not possible. Contradiction! Therefore, $(W,\lambda)$ is irreducible.

(9) Suppose $(V,\rho) \in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{H}}$. Then it belongs to $\mathrm{Rep}_{-}(\mathfrak{g},\mathbb{C})$, so let $(W,\lambda) := s_{-}(V,\rho) \in \mathrm{Rep}(\mathfrak{g},\mathbb{H})$. We have that $e_{-}(W,\lambda) \simeq (V,\rho)$, so

$$r_{\mathbb{C}}^{\mathbb{H}}(W,\lambda) = r_{-} \circ e_{-}(W,\lambda) \simeq r_{-}(V,\rho) = (V,\rho).$$

Thus,

$$e_{\mathbb{C}}^{\mathbb{H}}(V,\rho) \simeq e_{\mathbb{C}}^{\mathbb{H}} \circ r_{\mathbb{C}}^{\mathbb{H}}(W,\lambda) = (W,\lambda) \oplus (W,\lambda).$$

Let us decompose $(W,\lambda)$ as $(W,\lambda) = \bigoplus_{j=1}^{t}(W_j,\lambda_j)$, where $t \geq 1$ and $(W_j,\lambda_j) \in \mathrm{Irr}(\mathfrak{g},\mathbb{H})$. We see that

$$(V,\rho) \simeq \bigoplus_{j=1}^{t} r_{\mathbb{C}}^{\mathbb{H}}(W_j,\lambda_j).$$

As $(V,\rho)$ is irreducible, we have $t = 1$, so $(W,\lambda) \in \mathrm{Irr}(\mathfrak{g},\mathbb{H})_{\mathbb{H}}$, proving the first part.

Let $(U,\lambda) := r_{\mathbb{R}}^{\mathbb{C}}(V,\rho)$. If we prove that $(U,\lambda)$ is irreducible, then we have proven the result. Suppose not, then let us decompose it as $(U,\lambda) = \bigoplus_{j=1}^{t}(U_j,\lambda_j)$, where $t \geq 2$ and $(U_j,\lambda_j) \in \mathrm{Irr}(\mathfrak{g},\mathbb{R})$. We see that as $(V,\rho)$ is of quaternionic type,

$$(V,\rho) \oplus (V,\rho) = e_{\mathbb{R}}^{\mathbb{C}} \circ r_{\mathbb{R}}^{\mathbb{C}}(V,\rho) = \bigoplus_{j=1}^{t} e_{\mathbb{R}}^{\mathbb{C}}(U_j,\lambda_j).$$

As $t \geq 2$ and $(V,\rho)$ is irreducible, we see that $t = 2$ and $e_{\mathbb{R}}^{\mathbb{C}}(U_j,\lambda_j) = (V,\rho)$ for $j = 1,2$. We know from above that if $(U_j,\rho_j)$ is of quaternionic or complex type, then $(V,\rho) = e_{\mathbb{R}}^{\mathbb{C}}(W_j,\rho_j)$ is not irreducible, but it is. Thus, $(U_j,\rho_j)$ must be of real type. But then $(V,\rho) = e_{\mathbb{R}}^{\mathbb{C}}(U_j,\rho_j)$ is of real and quaternionic type, which is not possible. Contradiction! Therefore, $(U,\lambda)$ is irreducible. □

**Corollary 2.11.** *For all $(V,\rho) \in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{R}}$, $(V,\rho) \oplus (\overline{V},\overline{\rho})$ with $(V,\rho) \in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{C}}$, and $(V,\rho)^{\oplus 2}$ with $(V,\rho) \in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{H}}$, we can choose real matrices (of the same dimension) inducing irreducible real representations whose extensions are the above representations. Furthermore, all irreducible real representations are constructed this way.*

*For all $(V,\rho)^{\oplus 2}$ with $(V,\rho) \in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{R}}$, $(V,\rho)\oplus(\overline{V},\overline{\rho})$ with $(V,\rho) \in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{C}}$, and $(V,\rho) \in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{H}}$, we can choose quaternionic matrices (of half the dimension) that give rise to irreducible real representations whose restrictions give the above representations. Furthermore, all irreducible quaternionic representations are constructed this way.*

*Proof.* If we start with an irreducible real representation $(U,\rho)$, then, by Proposition 2.10, its extension is of the form in the statement of the proposition. Similarly, if we start with an irreducible quaternionic representation $(W,\rho)$, then its restriction is of the form in the statement of the proposition. Thus, it remains to show that for every one of the above representations, we can find an irreducible real or quaternionic representation whose extension or restriction, respectively, gives back the representation.

If we take $(V,\rho) \in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{R}}$, then as this representation is of real type and irreducible, let $(U,\lambda) := s_+(V,\rho)$. Consider

$$e_{\mathbb{R}}^{\mathbb{C}}(U,\lambda) = r_+ \circ e_+ \circ s_+(V,\rho) \simeq r_+(V,\rho) = (V,\rho).$$

Thus, $(U,\lambda)$ must be irreducible and its extension is $(V,\rho)$.

If we take $(V,\rho)\oplus(\overline{V},\overline{\rho})$ with $(V,\rho) \in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{C}}$, then as $(V,\rho)$ is of complex type and irreducible, let $(U,\lambda) := r_{\mathbb{R}}^{\mathbb{C}}(V,\rho)$. By Proposition 2.10, we have that $(U,\lambda)$ is irreducible (and of complex type). Furthermore, we have that

$$e_{\mathbb{R}}^{\mathbb{C}}(U,\lambda) = e_{\mathbb{R}}^{\mathbb{C}} \circ r_{\mathbb{R}}^{\mathbb{C}}(V,\rho) = (V,\rho) \oplus (\overline{V},\overline{\rho}).$$

Hence, we have an irreducible real representation whose extension is $(V,\rho) \oplus (\overline{V},\overline{\rho})$.

If we take $(V,\rho)\oplus(V,\rho)$ with $(V,\rho) \in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{H}}$, then as $(V,\rho)$ is of quaternionic type and irreducible, let $(U,\lambda) := r_{\mathbb{R}}^{\mathbb{C}}(V,\rho)$. By Proposition 2.10, we have that $(U,\lambda)$ is irreducible (and of quaternionic type). Furthermore, we have that

$$e_{\mathbb{R}}^{\mathbb{C}}(U,\lambda) = e_{\mathbb{R}}^{\mathbb{C}} \circ r_{\mathbb{R}}^{\mathbb{C}}(V,\rho) = (V,\rho) \oplus (V,\rho).$$

Hence, we have an irreducible real representation whose extension is $(V,\rho) \oplus (V,\rho)$.

If we take $(V,\rho) \in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{H}}$, then as this representation is of quaternionic type and irreducible, let $(W,\lambda) := s_-(V,\rho)$. Consider

$$r_{\mathbb{C}}^{\mathbb{H}}(W,\lambda) = r_- \circ e_- \circ s_-(V,\rho) \simeq r_-(V,\rho) = (V,\rho).$$

Thus, $(W,\lambda)$ must be irreducible and its restriction is $(V,\rho)$.

If we take $(V,\rho)\oplus(\overline{V},\overline{\rho})$ with $(V,\rho)\in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{C}}$, then as $(V,\rho)$ is of complex type and irreducible, let $(W,\lambda):=e_{\mathbb{C}}^{\mathbb{H}}(V,\rho)$. By Proposition 2.10, we have that $(W,\lambda)$ is irreducible (and of complex type). Furthermore, we have that

$$r_{\mathbb{C}}^{\mathbb{H}}(W,\lambda)=r_{\mathbb{C}}^{\mathbb{H}}\circ e_{\mathbb{C}}^{\mathbb{H}}(V,\rho)=(V,\rho)\oplus(\overline{V},\overline{\rho}).$$

Hence, we have an irreducible quaternionic representation whose restriction is $(V,\rho)\oplus(\overline{V},\overline{\rho})$.

If we take $(V,\rho)\oplus(V,\rho)$ with $(V,\rho)\in \mathrm{Irr}(\mathfrak{g},\mathbb{C})_{\mathbb{R}}$, then as $(V,\rho)$ is of real type and irreducible, let $(W,\lambda):=e_{\mathbb{C}}^{\mathbb{H}}(V,\rho)$. By Proposition 2.10, we have that $(W,\lambda)$ is irreducible (and of real type). Furthermore, we have that

$$r_{\mathbb{C}}^{\mathbb{H}}(W,\lambda)=r_{\mathbb{C}}^{\mathbb{H}}\circ e_{\mathbb{C}}^{\mathbb{H}}(V,\rho)=(V,\rho)\oplus(V,\rho).$$

Hence, we have an irreducible quaternionic representation whose restriction is $(V,\rho)\oplus(V,\rho)$. ∎

Note that the previous corollary tells us exactly how to get the irreducible real and quaternionic representations from the complex ones.

The next proposition gives us a simple characterization of the sets $\mathrm{Irr}(\mathfrak{g},\mathbb{R})_L$. It tells us that the endomorphism algebra determines the type of an irreducible real representation.

**Theorem 2.12.** *The endomorphism algebra* $\mathrm{Hom}_{\mathfrak{g}}(U,U)$ *of* $(U,\rho)\in\mathrm{Irr}(\mathfrak{g},\mathbb{R})$ *is isomorphic to* $L$ *if and only if* $(U,\rho)\in\mathrm{Irr}(\mathfrak{g},\mathbb{R})_L$.

*Proof.* Every $0\neq\phi\in\mathrm{Hom}_{\mathfrak{g}}(U,U)$ is invertible by Schur's Lemma. Hence, the endomorphism algebra is a division algebra over $\mathbb{R}$, so it is isomorphic to $\mathbb{R},\mathbb{C},\mathbb{H}$.

Suppose that $\mathrm{Hom}_{\mathfrak{g}}(U,U)\simeq\mathbb{C}$. Then $\mathbb{C}$ acts on $(U,\rho)$ so $(U,\rho)=r_{\mathbb{R}}^{\mathbb{C}}(V,\lambda)$ for some irreducible $(V,\lambda)$ (if $(V,\lambda)$ were not irreducible, then so too would be its restriction, $(U,\lambda)$). If $(V,\rho)$ is of real type, then Proposition 2.10 tells us that $(U,\rho)=r_{\mathbb{R}}^{\mathbb{C}}(V,\lambda)$ is not irreducible, but it is. Thus, $(U,\rho)$ is of complex or quaternionic type. If it were of quaternionic type, then $e_{\mathbb{C}}^{\mathbb{H}}(V,\rho)=(W,\mu)\oplus(W,\mu)$ for some $(W,\mu)\in\mathrm{Irr}(\mathfrak{g},\mathbb{H})_{\mathbb{H}}$. But then, as $(V,\lambda)$ is of quaternionic type

$$r_{\mathbb{C}}^{\mathbb{H}}(W,\mu)\oplus r_{\mathbb{C}}^{\mathbb{H}}(W,\mu)=r_{\mathbb{C}}^{\mathbb{H}}\circ e_{\mathbb{C}}^{\mathbb{H}}(V,\lambda)=(V,\lambda)\oplus(V,\lambda).$$

Hence, $(V,\lambda)=r_{\mathbb{C}}^{\mathbb{H}}(W,\mu)$, so $(U,\rho)=r_{\mathbb{R}}^{\mathbb{H}}(W,\mu)$. Thus, $\mathbb{H}$ is contained in $\mathrm{Hom}_{\mathfrak{g}}(U,U)$, which it is not. Thus, $(V,\lambda)$ is of complex type, so $(U,\rho)$ is too.

Suppose that $\mathrm{Hom}_{\mathfrak{g}}(U,U)\simeq\mathbb{H}$. Then $\mathbb{H}$ acts on $(U,\rho)$ so $(U,\rho)=r_{\mathbb{R}}^{\mathbb{H}}(W,\mu)$, for some $(W,\mu)$ irreducible (for the same reason as above). From Proposition 2.10, we see that as $r_{\mathbb{R}}^{\mathbb{H}}=r_{\mathbb{R}}^{\mathbb{C}}\circ r_{\mathbb{C}}^{\mathbb{H}}$, if $(W,\mu)$ is of real or complex type, then $(U,\rho)=r_{\mathbb{R}}^{\mathbb{H}}(W,\mu)$

is not irreducible, but it is. Thus, $(W, \mu)$ is of quaternionic type. Hence, $r_{\mathbb{C}}^{\mathbb{H}}(W, \mu)$ is of quaternionic type and irreducible, so $(U, \rho) = r_{\mathbb{R}}^{\mathbb{C}} \circ r_{\mathbb{C}}^{\mathbb{H}}(W, \mu)$ is also of quaternionic type.

Finally, suppose that $\mathrm{Hom}_{\mathfrak{g}}(U, U) \simeq \mathbb{R}$. Then $(U, \rho)$ is not of the form $r_{\mathbb{R}}^{\mathbb{C}}(V, \rho)$ for $(V, \rho)$ irreducible and of complex or quaternionic type, otherwise $\mathbb{C} \subseteq \mathrm{Hom}_{\mathfrak{g}}(U, U)$. Thus, we must have, by Theorem 2.2, that $(U, \rho)$ is of real type. □

# 3 Classifying Representations

In this section, we determine which irreducible, complex representations of $\mathfrak{sp}(1)$ are of real, complex, and quaternionic type. We find that this Lie algebra admits no complex type representations, so all their representations are self-dual.

The irreducible, complex representation $(V_n, \rho_n)$ of $\mathfrak{su}(2)$ is explicitly given as follows.

**Definition 3.1.** *Let $V_n$ be the space of homogeneous polynomials in $X, Y$ of degree $n-1$. Note that $\dim(V_n) = n$. We view polynomials as functions on $\mathbb{C}^2$. Consider $\rho_n \colon \mathfrak{su}(2) \to \mathfrak{gl}(V_n)$ given by*

$$\rho_n(\upsilon) := \begin{bmatrix} X & Y \end{bmatrix} \upsilon \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix}.$$

We see that $\rho_n(\upsilon) \colon V_n \to V_n$.

**Lemma 3.2.** *The pair $(V_n, \rho_n)$ is a representation of $\mathfrak{sp}(1)$.*

*Proof.* We see that $\rho_n$ is linear, so we need only check it is a Lie algebra homomorphism. We see

$$[\rho_n(\upsilon), \rho_n(\tau)] = \left[ \begin{bmatrix} X & Y \end{bmatrix} \upsilon \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix}, \begin{bmatrix} X & Y \end{bmatrix} \tau \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix} \right].$$

We see that

$$\begin{aligned}
\begin{bmatrix} X & Y \end{bmatrix} \upsilon \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix} \begin{bmatrix} X & Y \end{bmatrix} \tau \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix} &= \begin{bmatrix} X & Y \end{bmatrix} \upsilon \left( \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix} \begin{bmatrix} X & Y \end{bmatrix} \right) \tau \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix} \\
&\quad + \begin{bmatrix} X & Y \end{bmatrix} \upsilon \begin{bmatrix} X\frac{\partial}{\partial X} & Y\frac{\partial}{\partial X} \\ X\frac{\partial}{\partial Y} & Y\frac{\partial}{\partial Y} \end{bmatrix} \tau \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix} \\
&= \begin{bmatrix} X & Y \end{bmatrix} \upsilon\tau \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix} \\
&\quad + \begin{bmatrix} X & Y \end{bmatrix} \upsilon \begin{bmatrix} X\frac{\partial}{\partial X} & Y\frac{\partial}{\partial X} \\ X\frac{\partial}{\partial Y} & Y\frac{\partial}{\partial Y} \end{bmatrix} \tau \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix}
\end{aligned}$$

Substituting, we see that

$$\begin{aligned}
[\rho_n(\upsilon), \rho_n(\tau)] = \begin{bmatrix} X & Y \end{bmatrix} [\upsilon, \tau] \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix} \\
+ \begin{bmatrix} X & Y \end{bmatrix} \left( \upsilon \begin{bmatrix} X\frac{\partial}{\partial X} & Y\frac{\partial}{\partial X} \\ X\frac{\partial}{\partial Y} & Y\frac{\partial}{\partial Y} \end{bmatrix} \tau - \tau \begin{bmatrix} X\frac{\partial}{\partial X} & Y\frac{\partial}{\partial X} \\ X\frac{\partial}{\partial Y} & Y\frac{\partial}{\partial Y} \end{bmatrix} \upsilon \right) \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix}.
\end{aligned}$$

Computing the final term for arbitrary $\upsilon, \tau \in \mathfrak{su}(2)$, we see that it vanishes. Thus, we are left with

$$[\rho_n(\upsilon), \rho_n(\tau)] = \rho_n([\upsilon, \tau]).$$

That is, we have a representation. □

**Note 3.3.** *Using the basis of* $\mathfrak{su}(2)$ *given by* $\upsilon_1 := \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $\upsilon_2 := \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, *and* $\upsilon_3 := \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, *we find*

$$\begin{aligned}
\rho_n(\upsilon_1) &= \frac{i}{2}X\frac{\partial}{\partial X} - \frac{i}{2}Y\frac{\partial}{\partial Y}, \\
\rho_n(\upsilon_2) &= \frac{1}{2}X\frac{\partial}{\partial Y} - \frac{1}{2}Y\frac{\partial}{\partial X}, \\
\rho_n(\upsilon_3) &= \frac{i}{2}X\frac{\partial}{\partial Y} + \frac{i}{2}Y\frac{\partial}{\partial X}.
\end{aligned}$$

**Lemma 3.4.** *This representation is irreducible.*

*Proof.* Consider the basis $\{X^{n-1}, \ldots, X^j Y^{n-1-j}, \ldots, Y^{n-1}\}$ of $V_n$. We compute the Casimir operator in this basis and show that it is $\frac{n^2-1}{4} I_n$. Note that as $\mathfrak{sp}(1)$ is rank one, the Casimir operator determines the representation completely.

We see that for $j \in \{0, \ldots, n-1\}$, we have

$$\begin{aligned}
\rho_n(\upsilon_1) X^j Y^{n-1-j} &= \frac{i}{2}\left(2j - n + 1\right) X^j Y^{n-1-j}, \\
\rho_n(\upsilon_2) X^j Y^{n-1-j} &= \frac{1}{2}(n-1-j) X^{j+1} Y^{n-2-j} - \frac{1}{2} j X^{j-1} Y^{n-j}, \\
\rho_n(\upsilon_3) X^j Y^{n-1-j} &= \frac{i}{2}(n-1-j) X^{j+1} Y^{n-2-j} + \frac{i}{2} j X^{j-1} Y^{n-j}.
\end{aligned}$$

Note that the coefficients vanish when we would get elements not in our basis.

Written as matrices in the aforementioned basis,

$$\rho_n(\upsilon_1) = \frac{i}{2}\begin{bmatrix} n-1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -n+1 \end{bmatrix},$$

$$\rho_n(\upsilon_2) = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ -n+1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & n-1 \\ 0 & \cdots & 0 & -1 & 0 \end{bmatrix},$$

$$\rho_n(\upsilon_3) = \frac{i}{2}\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ n-1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & n-1 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

The Casimir operator $\phi = -\sum_{j=1}^{3} \rho_n(\upsilon_j)^2$ is then exactly $\frac{n^2-1}{4} I_n$, the Casimir operator of the irreducible, complex $n$-dimensional representation. □

Following Itzkowitz et al., we now find real and quaternionic structures for different values of $n$ [IRS91].

**Proposition 3.5.** *For $n$ even, $(V_n, \rho_n)$ is quaternionic type. For $n$ odd, it is real type.*

**Note 3.6.** *As all irreducible, complex representations are of real or quaternionic type, they are all self-dual. In particular, we have*

$$\begin{aligned}
\operatorname{Irr}(\mathfrak{sp}(1), \mathbb{C})_{\mathbb{R}} &= \{(V_n, \rho_n) \mid n \text{ odd}\}, \\
\operatorname{Irr}(\mathfrak{sp}(1), \mathbb{C})_{\mathbb{C}} &= \emptyset, \\
\operatorname{Irr}(\mathfrak{sp}(1), \mathbb{C})_{\mathbb{H}} &= \{(V_n, \rho_n) \mid n \text{ even}\}.
\end{aligned}$$

*Thus, by Corollary 2.11, for $n$ odd or divisible by four, there is a unique irreducible real $n$-representation $(\mathbb{R}^n, \varrho_n)$ whose complexification is $(V_n, \rho_n)$ when $n$ is odd or $(V_{n/2}, \rho_{n/2})^{\oplus 2})$ when $n$ is divisible by four.*

*Moreover, when restricting the scalars of an irreducible quaternionic representation to $\mathbb{C}$, the complex representation is isomorphic to $(V_n, \rho_n)$ for some $n$ even or $(V_n, \rho_n)^{\oplus 2}$ for some $n$ odd.*

*Proof.* Define $J_n \colon V_n \to V_n$ as follows. Given $P(X, Y) = \sum_{j=0}^{n-1} a_j X^j Y^{n-1-j}$, let $\overline{P}(X, Y) := \sum_{j=0}^{n-1} \overline{a_j} X^j Y^{n-1-j}$. Then define $J_n(P(X, Y)) := \overline{P}(-Y, X)$, which simplifies to $J_n(P(X, Y)) = \sum_{j=0}^{n-1} \overline{a_j}(-Y)^j X^{n-1-j}$. Note that $J_n$ is conjugate-linear. We see that $J_n X^j Y^{n-1-j} = (-1)^j X^{n-1-j} Y^j$. Writing $J_n$ in the same basis as the $\rho_n(\upsilon_i)$, we find

$$J_n = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \vdots \\ (-1)^{n-1} & 0 & \cdots & 0 \end{bmatrix}.$$

Recalling that $J_n$ is conjugate-linear, we see that $J_n$ commutes with the $\rho_n(\upsilon_i)$. Furthermore, we see that

$$J_n^2(P(X, Y)) = J_n(\overline{P}(-Y, X)) = P(-X, -Y) = (-1)^{n-1} P(X, Y),$$

recalling that $P$ is a homogeneous degree $n - 1$ polynomial. Therefore, if $n$ is even, then $J_n$ is a quaternionic structure and if $n$ is odd, it is a real structure. □

Consider Corollary 2.11. As we have no complex type representations, we see that for $(V_n, \rho_n)$ with $n$ odd and $(V_n, \rho_n) \oplus (V_n, \rho_n)$ with $n$ even, we can choose real matrices (of the same dimension) inducing irreducible real representations and these are all the irreducible real representations. Additionally, for $(V_n, \rho_n)$ with $n$ even and $(V_n, \rho_n) \oplus (V_n, \rho_n)$ with $n$ odd, we can choose quaternionic matrices (of half the dimension) inducing irreducible quaternionic representations and these are all the irreducible quaternionic representations.

# References

[BTD85] T. Bröcker and T. Tom Dieck. *Representations of Compact Lie Groups*. Graduate Texts in Mathematics. Springer Berlin Heidelberg, Berlin, 1st edition, 1985. doi:10.1007/978-3-662-12918-0. 1

[IRS91] G. Itzkowitz, S. Rothman, and H. Strassberg. A note on the real representations of SU(2,C). *Journal of Pure and Applied Algebra*, 69(3):285–294, 1991. doi:10.1016/0022-4049(91)90023-U. 23